Mathematics 120 Final Exam – F. Goodman
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Do six of the following problems problems. In case you attempt more than six, indicate clearly which six are to be graded. Responses will be judged for accuracy, clarity and coherence.

1. Let G be a finite group and H a subgroup. Let $a, b \in G$. Show that either $aH = bH$ or $aH \cap bH = \emptyset$. Show that $\#(aH) = \#(bH)$. Conclude that $\#H\ \#(G/H) = \#G$.

2. Let x, y be elements of a group G such that xy is in the center of G. Show that x and y commute.

3. Let G be a group, $N \trianglelefteq G$ a normal subgroup and $A \leq G$ any subgroup. Show that $AN/N \cong A/(A \cap N)$. If G is finite, conclude that $\#AN = \dfrac{\#A\ \#N}{\#(A \cap N)}$.

4. What are all the normal subgroups of S_4?

5. Let G be a group and let H be a subgroup of finite index $[G : H] = n$. Consider the transitive action of G by left multiplication on G/H; this yields a homomorphism of G into $\mathrm{Sym}(G/H) \cong S_n$.

- **(a)** Describe the kernel of this homomorphism. (Hint: It must be normal, and it must have something to do with H.)
- **(b)** Describe the kernel explicitly when $G = S_4$ and H is the group of order 8 generated by (1234) and (12)(34). Show that the resulting homomorphism of S_4 into S_3 is surjective.

6. What are all the 2- Sylow subgroups of S_4? If P is a 2-Sylow subgroup, show that the normalizer $N_{S_4}(P)$ of P is equal to P. (Hint: Consider the index of $N_{S_4}(P)$ and the index of P.)

7. Show that any group of order 35 is cyclic. Show that a group of order 70 contains a normal subgroup of order 35, but there exist non-abelian groups of order 70.