Mathematics 120 Midterm Exam I – F. Goodman
October, 2994
Version 1

Do all of problems 1-4 and one of problems 5-6. Responses will be judged for accuracy, clarity and coherence.

1. Give the definition of a *normal* subgroup of a group. (Note: you have to define *normal*; you do not have to define *group* or *subgroup*.) Give an example of a normal subgroup of an non-abelian group and prove that your subgroup is in fact normal. Give an example of a non-normal subgroup of a group, and prove that your subgroup is not normal.

2. Show that the quotient of a group by a normal subgroup is a group.

3. State and prove the homomorphism theorem (a.k.a. the first isomorphism theorem).

4. **(a)** Let

$$\varphi : \mathbb{Z}_n \longrightarrow W$$

be a surjective group homomorphism. Show that W is cyclic of order m, where m divides n. Moreover, show there is an isomorphism

$$\psi : W \longrightarrow \mathbb{Z}_m$$

such that $\psi \circ \varphi([j]_n) = [j]_m$ for all j.

(b) Conclude that the kernel of φ is the set of $[j]_n$ such that m divides j. In particular, the kernel is determined by the size of W.

(c) We proved in class (using Euclid's algorithm) that for each k dividing n, there is a unique subgroup of $\mathbb{Z}_n$ of order k. Recover this result by considering the quotient group and the quotient map. That is, suppose N is a subgroup of $\mathbb{Z}_n$ of order k. Let $m = n/k$. By considering the quotient map $\pi : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n/N$, conclude that $N = \{[j]_n : m \text{ divides } j\}$

5. We showed in an exercise that there are exactly two groups of order 4 up to isomorphism. We did this just using multiplication tables, before we knew Lagrange's theorem. Now recover this result with the help of Lagrange's theorem. Namely, suppose G is a group of order 4 which is not cyclic. Conclude that every non-identity element has order 2. Now show that the multiplication of G is completely determined.

6. Suppose G is a non-empty set with an associative multiplication with the property that for every $a, b \in G$, there is an x such that $ax = b$ and there is a y such that $ya = b$. Conclude that G is a group.