22M:121 Spring 2005
Assignment No. 9

Some of these problems have been taken from old Ms Comps. In that spirit, you will have to choose and solve any 6 problems from the following list.
Due Friday April 8^{th}.

1. Let V be a finite dimensional vector space over a field K, $V \neq \{0\}$, and let $A, B : V \longrightarrow V$ be $K-$linear maps.

   a) Show that the eigenvalues of AB are the same as the eigenvalues of BA.

   b) Suppose A is invertible and λ is an eigenvalue of A. Prove that $A \neq 0$ and that λ^{-1} is an eigenvalue of A^{-1}.

2. Determine whether or not the two matrices
$$A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & -1 \\ -4 & 0 & 3 \end{bmatrix} \textit{ and } B = \begin{bmatrix} 5 & -8 & 4 \\ 6 & -11 & 6 \\ 6 & -12 & 7 \end{bmatrix}$$
are similar over Q.

3. Given the matrix
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$
prove that there is a 3×3 matrix C such that
$$CAC^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4. Let A be an $n \times n$ matrix such that the sum of the elements in any row is 1. Show that A has an eigenvalue equal to 1.

5. Fix a, b, c in a field K so that $a \neq 0$.

   Show that all the matrices $A_{a,b,c}$ given by
$$A_{a,b,c} = \begin{bmatrix} 3 & 0 & 0 \\ a & 3 & 0 \\ b & c & -2 \end{bmatrix}$$
are similar.

6. **a)** Find the rational canonical form of $A = \begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix} \in M_2(Q)$.

   b) Prove that two 2×2 matrices over a field F which are not scalar matrices (of the form αId for some $\alpha \in F$) are similar if and only if they have the same characteristic polynomial.

7. Find all similarity classes of 6×6 matrices over C with characteristic polynomial $(x^4 - 1)(x^2 - 1)$.
8. Find all similarity classes of 6×6 matrices over Q with minimal polynomial $(x+2)^2(x-1)$.
9. Prove that two 3×3 matrices over a field F are similar if and only if they have the same characteristic and same minimal polynomials.