Mathematics 121 Review for Midterm I
February, 2005

Definitions you should know:

1. Ring, subring, ring homomorphism, ideal.
2. Integral domain, Euclidean domain, unique factorization domain, field.
3. Unit, associate, irreducible element, prime element.
4. Maximal ideal, prime ideal, principal ideal.
5. Simple ring.
6. Ascending chain condition for ideals, for principal ideals.

Theorems you should be able to prove:

1. The substitution principle (6.2.5 in revised text); proof to include the computation that the proposed map is a homomorphism.
2. Extension of homomorphisms to polynomial rings (6.2.8).
3. The kernel of a ring homomorphism is an ideal.
4. A commutative ring with 1 is simple if, and only if, it is a field.
5. The ring of n–by–n matrices over a field is simple.
6. The intersection of any family of ideals is an ideal. The union of an increasing sequence of ideals is an ideal. The product and sum of ideals is an ideal.
7. Every ideal in $\mathbb{Z}$ or in $K[x]$ is principal. In general, every ideal in a Euclidean domain is principal.
8. The quotient ring and homomorphism theorems: Results 1, 4, 7, 8 , 9, 10 from section 6.3 (in the revised text).
9. Consider the following conditions on a nonzero, nonunit element p in an integral domain R:
   - pR is a maximal ideal.
   - pR is a prime ideal.
   - p is prime.
   - p is irreducible.

   (a) What implications hold concerning these conditions? Prove these implications.
   (b) If R is a UFD, what additional implication(s) hold. Prove these implications.
   (c) If R is a PID, what additional implication(s) hold. Prove these implications.
   Remark for the literal minded: Of course, you don't have to prove every possible implication. If A implies B and B implies C, you don't have to prove in addition that A implies C.
10. A PID is a UFD.
11. If R is a UFD, then $R[x]$ is a UFD.
12. R is a UFD if and only if R has the ACC for principal ideals and every irreducible in R is prime.
13. Eisenstein's criterion.

Be able to provide examples of:

1. A UFD that is not a PID.
2. An integral domain that is not a UFD.
3. An element in an integral domain with no irreducible factorization at all.
4. An element in an integral domain with non-unique irreducible factorizations.
5. An element in an integral domain that is irreducible but not prime.