Antiderivatives of trigonometric functions

1. Powers of $\sin(x)$ or $\cos(x)$

1.1. **Odd powers of $\cos(x)$.**

Antiderivatives of an odd power of $\cos(x)$ are handled by first splitting off one factor of $\cos(x)$, for example:

$$\int \cos^5(x)dx = \int \cos^4(x)\cos(x)dx.$$

The remaining even power of $\cos(x)$ is expressed in terms of $\sin(x)$ using the identity

$$\sin^2(x) + \cos^2(x) = 1,$$

for example

$$\begin{aligned}\int \cos^4(x)\cos(x)dx &= \int (\cos^2(x))^2 \cos(x)dx \\ &= \int (1-\sin^2(x))^2 \cos(x)dx\end{aligned}$$

One now makes the substitution $u = \sin(x)$, $du = \cos(x)dx$, to get

$$\begin{aligned}\int (1-\sin^2(x))^2 \cos(x)dx &= \int (1-u^2)^2 du \\ &= \int (1-2u^2+u^4)du.\end{aligned}$$

This is now straightforward to compute,

$$\begin{aligned}\int (1-2u^2+u^4)du &= u - (2/3)u^3 + (1/5)u^5 + c \\ &= \sin(x) - (2/3)\sin^3(x) + (1/5)\sin^5(x) + c.\end{aligned}$$

1.2. **Odd powers of $\sin(x)$.**

Odd powers of $\sin(x)$ are handled similarly for example:

$$\int \sin^5(x)dx = \int \sin^4(x)\sin(x)dx.$$

After splitting off one factor of $\sin(x)$, express the remaining even power of $\sin(x)$ in terms of $\cos(x)$ by using the identity

$$\sin^2(x) + \cos^2(x) = 1.$$

In our example,

$$\int \sin^4(x)\sin(x)dx = \int (\sin^2(x))^2 \sin(x)dx = \int (1-\cos^2(x))^2 \sin(x)dx$$

One now makes the substitution $u = \cos(x)$, $du = -\sin(x)dx$, to get

$$\int (1-\cos^2(x))^2 \sin(x)dx = -\int (1-u^2)^2\ du = -\int (1-2u^2+u^4)du.$$

1.3. **Even powers of** $\sin(x)$ **or** $\cos(x)$**.** Even powers of $\sin(x)$ or $\cos(x)$ must be handled in a different way. One uses the identities

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \quad \text{and} \quad \sin^2(x) = \frac{1-\cos(2x)}{2}.$$

Thus

$$\int \cos^4(x)dx = \int (\cos^2(x))^2 dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx.$$

Expanding this gives

$$\int (1/4)(1 + 2\cos(2x) + \cos^2(2x))dx$$
$$= (1/4)\int 1\ dx + (1/2)\int \cos(2x)\ dx + (1/4)\int \cos^2(2x)\ dx.$$

Note that this antidifferntiation involves various powers of $\cos(2x)$. The first two antiderivatives in the last line are elementary:

$$(1/4)\int 1\ dx = (1/4)x + c,$$

$$(1/2)\int \cos(2x)\ dx = (1/4)\sin(2x) + c.$$

The antiderivative

$$(1/4)\int \cos^2(2x)\ dx$$

has to be handled by repeating the procedure,

$$\begin{aligned}(1/4)\int \cos^2(2x)\ dx &= (1/4)\int \frac{1+\cos(4x)}{2}\ dx\\ &= (1/8)\int 1\ dx + (1/8)\int \cos(4x)\ dx\\ &= (1/8)x + (1/32)\sin(4x) + c.\end{aligned}$$

Adding up all the pieces, one gets

$$\begin{aligned}\int \cos^4(x)dx &= (1/4)x + (1/4)\sin(2x) + (1/8)x + (1/32)\sin(4x) + c\\ &= (3/8)x + (1/4)\sin(2x) + (1/32)\sin(4x) + c\end{aligned}$$

2. Products of powers of $\sin(x)$ and $\cos(x)$

Products of powers of $\sin(x)$ and $\cos(x)$ can be handled by the same techniques. If either $\sin(x)$ or $\cos(x)$ appears with an odd exponent, the technique for odd powers can be used. If both appear with even exponents, the technique for even powers is used.

Example 2.1.

$$\begin{aligned}\int \sin^3(x)\cos^2(x)\ dx &= \int \sin^2(x)\cos^2(x)\sin(x)\ dx\\ &= \int (1-\cos^2(x))\cos^2(x)\sin(x)\ dx\end{aligned}$$

Now make the substitution $u = \cos(x)$, $du = -\sin(x)dx$

$$\int (1-\cos^2(x))\cos^2(x)\sin(x)\ dx = -\int (1-u^2)u^2 du$$
$$= \int u^4 - u^2 du = \frac{u^5}{5} - \frac{u^3}{3} + c$$
$$= \frac{\cos(x)^5}{5} - \frac{\cos(x)^3}{3} + c$$

Example 2.2.

$$\int \sin^2(x)\cos^2(x)\ dx = \int (1/4)(1-\cos(2x))(1+\cos(2x))\ dx$$
$$= (1/4)\int (1-\cos^2(2x))\ dx$$

Now repeat the trick

$$(1/4)\int (1-\cos^2(2x))\ dx = (1/4)\int 1-(1/2)(1+\cos(4x))\ dx$$
$$= (1/4)\int (1/2)-(1/2)\cos(4x)\ dx$$
$$= (1/4)[(1/2)x-(1/8)\sin(4x)]+c$$
$$= (1/8)-(1/32)\sin(4x)+c.$$