Cellularity and the Jones basic construction

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Introduction

The goal of this work is to study certain finite dimensional algebras that arise in invariant theory, knot theory, subfactors, QFT, and statistical mechanics.

The algebras in question have parameters; for generic values of the parameters, they are semisimple, but it is also interesting to study non–semisimple specializations. It turns out that certain ideas from the semisimple world— specifically, the Jones basic construction — are still useful in the non–semisimple case.

The Jones basic construction

There are two themes in this work: “Cellularity" and “the Jones basic construction." Cellularity means something specific, and it will be defined.

What the Jones basic construction means in the general context we consider is not exactly clear. When it does make sense, the construction is a machine which, given a pair of algebras $\mathbf{1} \in A \subseteq B$, will produce a third algebra J, with $A \subseteq B \subseteq J$.

If A and B are split semisimple over a field F, then it is clear what J should be, namely $J = \mathrm{End}(B_A)$.

Suppose now that A and B are not only split s.s., but also that we have an F–valued trace on B, which is faithful on B and has faithful restriction to A. (Faithful means that the bilinear form $(x, y) = \mathrm{tr}(xy)$ is non–degenerate.)

Basic construction, cont.

In the case just described, we have a unital A–A bimodule map $\varepsilon : B \to A$ determined by $\mathrm{tr}(ba) = \mathrm{tr}(\varepsilon(b)a)$ for $b \in B$ and $a \in A$. Then we also have

$$\mathrm{End}(B_A) = B\varepsilon B \cong B \otimes_A B,$$

where the latter isomorphism is as B–B bimodules

Now consider three successive **BMW algebras** $A_{n-1} \subseteq A_n \subseteq A_{n+1}$. Then A_{n+1} contains an essential idempotent e_n, and we would like to understand the ideal $J_{n+1} = A_{n+1} e_n A_{n+1}$.

Basic construction, slide 3

Suppose (for the moment) that we work over $\mathbb{C}$ and the parameters of the BMW algebras are chosen generically. Then all the algebras in sight are s.s. and

$$J_{n+1} \cong \mathrm{End}((A_n)_{A_{n-1}}) \cong A_n \otimes_{A_{n-1}} A_n.$$

That is, J_{n+1} is the basic construction for $A_{n-1} \subseteq A_n$.

Does some part of this persist in the non–s.s. case? Now, work over the generic integral ground ring for the BMW algebras; i.e. work with the “integral form" of the BMW algebras. Now J_{n+1} is no longer even a unital algebra and A_n is not projective as an A_{n-1} module. Nevertheless, it remains true that

$$J_{n+1} \cong A_n \otimes_{A_{n-1}} A_n.$$

This is what makes our treatment of cellularity for BMW algebras, and other similar algebras, work.

Cellularity

Cellularity is a concept due to Graham and Lehrer that is useful in the study of a number of important algebras: Hecke algebras, Brauer and BMW algebras, Schur algebras, etc.

I will give the definition, say what it is good for, and then make some general observations about cellularity, including a suggested variant on the definition, and a new basis free formulation.

What is cellularity?

Let A be an algebra with involution $*$ over an integral domain S. A is said to be cellular if there exists a finite partially ordered set $(\Lambda, \geq)$ and for each $\lambda \in \Lambda$, a finite index set $\mathcal{T}(\lambda)$, such that

- A has an S–basis $\{c^{\lambda}_{\mathfrak{s},\mathfrak{t}} : \lambda \in \Lambda; \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$.
- $(c^{\lambda}_{\mathfrak{s},\mathfrak{t}})^* = c^{\lambda}_{\mathfrak{t},\mathfrak{s}}$.
- For $a \in A$,

$$a\, c^{\lambda}_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{\mathfrak{r}} s_{\mathfrak{r}} c^{\lambda}_{\mathfrak{r},\mathfrak{t}},$$

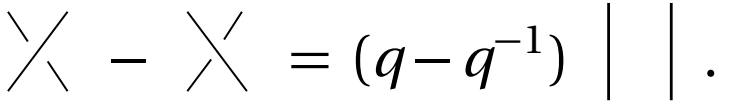
  modulo the span of basis elements $c^{\mu}_{\mathfrak{u},\mathfrak{v}}$, with $\mu > \lambda$, where the coefficients in the expansion depend only on a and $\mathfrak{s}$ and not on $\mathfrak{t}$.

Such a basis is called a cellular basis, and the whole apparatus $(\Lambda, \geq, \mathcal{T}(\lambda), \{c^{\lambda}_{\mathfrak{s},\mathfrak{t}}\})$ is called a cell datum.

Cellularity – Example

Definition 1

The Hecke algebra $H_n^S(q^2)$ over a ring S is the quotient of the braid group algebra over S by the Hecke skein relation:



Fact:
The Hecke algebras $H_n^S(q^2)$ are cellular, with $\Lambda = Y_n$, the set of Young diagrams with n boxes, ordered by *dominance*, and $\mathcal{T}(\lambda)$ the set of standard tableaux of shape λ. The cell modules are known as *Specht modules*. The cellular structure is due to Murphy.

See, for example, A. Mathas, *Iwahori-Hecke Algebras and Schur Algebras of the Symmetric Group*, AMS University Lecture Series.

What is cellularity?, cont.

It follows immediately from the definition of cellularity that

- For every order ideal Γ of Λ,

$$A(\Gamma) := \operatorname{span}\{c^{\lambda}_{\mathfrak{s},\mathfrak{t}} : \lambda \in \Gamma, \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\lambda)\}$$

  is a $*$–ideal of A.

  In particular, write A^{λ} for $A(\{\mu : \mu \geq \lambda\})$ and $\check{A}^{\lambda}$ for $A(\{\mu : \mu > \lambda\})$.

- For each $\lambda \in \Lambda$, there is an A–module Δ^{λ}, free as S–module, with basis $\{c^{\lambda}_{\mathfrak{t}} : \mathfrak{t} \in \mathscr{T}(\lambda)\}$, such that the map $\alpha^{\lambda} : A^{\lambda}/\check{A}^{\lambda} \to \Delta^{\lambda} \otimes_R (\Delta^{\lambda})^*$ defined by $\alpha^{\lambda} : c^{\lambda}_{\mathfrak{s},\mathfrak{t}} + \check{A}^{\lambda} \mapsto c_{\mathfrak{s}} \otimes (c^{\lambda}_{\mathfrak{t}})^*$ is an A–A bimodule isomorphism.

What is this good for?

- The modules Δ^λ are called *cell modules*. When the ground ring is a field, and the algebra A is semisimple, these are exactly the simple modules.
- In general, general, Δ^λ has a canonical bilinear form. With $\mathrm{rad}(\lambda)$ the radical of this form, and with the ground ring a field, $\Delta^\lambda/\mathrm{rad}(\lambda)$ is either zero or simple, and all simples are of this form.

So cellularity gives an approach to finding all the simple modules. It's also useful for investigating the block structure, etc.

Proposed weakening of the definition

I propose weakening the definition of cellularity, replacing

- $(c^{\lambda}_{\mathfrak{s},\mathfrak{t}})^* = c^{\lambda}_{\mathfrak{t},\mathfrak{s}}$

by

- $(c^{\lambda}_{\mathfrak{s},\mathfrak{t}})^* \equiv c^{\lambda}_{\mathfrak{t},\mathfrak{s}}$ modulo $\check{A}^{\lambda}$.

Disadvantages: none that I know of.

All the results of Graham & Lehrer are still valid. Moreover, the weakened definition is equivalent to the original definition if 2 is invertible in the ground ring, so we haven't lost much.

Advantages of the weaker definition

- Main advantage is graceful treatment of extensions with weaker definition: If J is a “cellular ideal" in A and A/J is cellular, then A is cellular.
- Cellular algebras can have many different cellular bases yielding the “same cellular structure," i.e. same ideals, same cell modules. Suppose one has a cellular algebra with cell modules Δ^λ. With the weaker definition, one easily sees that *any* collection of bases of the cell modules can be lifted (globalized) to a cellular basis of the algebra, via the maps $\alpha^\lambda : A^\lambda/\check{A}^\lambda \to \Delta^\lambda \otimes_R (\Delta^\lambda)^*$

Cellularity – basis free formulation

With the weakened definition of cellularity, one gets a basis free formulation of cellularity (improving on work of König & Xi). In the following, a "Λ cell net" is a map from order ideals of Λ to $*$–ideals of A, with several natural properties. Denote the map $\Gamma \mapsto J(\Gamma)$.

The most important properity is: for each $\lambda \in \Lambda$, there exists an A–module M^λ, free as S–module, such that $J(\Gamma)/J(\Gamma') \cong M^\lambda \otimes_S (M^\lambda)^*$, as A–A bimodules, whenever $\Gamma \setminus \Gamma' = \{\lambda\}$.

Proposition 2

Let A be an R–algebra with involution, and let $(\Lambda, \geq)$ be a finite partially ordered set. Then A has a cell datum **with partially ordered set Λ** *if, and only if, A has a Λ–cell net. Moreover, in this case, the M^λ are the cell modules for the cellular structure.*

Our objects of study

We develop a framework for studying cellularity for several important examples of **pairs of towers of algebras**,

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \quad \text{and} \quad Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \dots$$

such as

- $A_n =$ Brauer algebra, $Q_n =$ symmetric group algebra.
- $A_n =$ BMW algebra, $Q_n =$ Hecke algebra.
- $A_n =$ cyclotomic BMW algebra, $Q_n =$ cyclotomic Hecke algebra.
- $A_n =$ partition algebra, $Q_n =$ “stuttering" sequence of symmetric group algebras.
- etc.

Some properties of the examples

1. There is a generic (integral) ground ring R for A_n (independent of n). Every instance of A_n over a ground ring S is a specialization: $A_n^S = A_n^R \otimes_R S$.
2. With F the field of fractions of R, the algebra A_n^F is semisimple.
3. A_n contains an essential idempotent e_{n-1}. Set $J_n = A_n e_{n-1} A_n$. Then: $Q_n \cong A_n/J_n$, and J_{n+1}^F is isomorphic to a Jones basic construction for the pair $A_{n-1}^F \subset A_n^F$.

All this results from a method due to Wenzl in the 80's for showing the generic semisimplicity and determining the generic branching diagram of the tower $(A_n^F)_{n\geq 0}$. (Wenzl applied this to Brauer and BMW algebras.)

What we did

In this work we have found a cellular analogue of Wenzl's construction, which gives a uniform proof of cellularity of the algebras A_n in the examples, with additional desirable features.

Note that for results about cellularity, it suffices for us to work over the generic ground ring R, since cellularity is preserved under specialization.

Coherence of cellular structures

It is a general principle that representation theories of the Hecke algebras $H_n(q)$ or of the symmetric group algebras KS_n should be considered all together, that induction/restriction between H_n and H_{n-1} plays a role in building up the representation theory.

Coherence of cellular structures is the cellular version of this principle.

Coherence and strong coherence

Definition 3

A sequence $(A_n)_{n\geq 0}$ *of cellular algebras, with cell data* $(\Lambda_n, \ldots)$ *is* coherent *if for each* $\mu \in \Lambda_n$*, the restriction of* Δ^μ *to* A_{n-1} *has a filtration*

$$\mathrm{Res}(\Delta^\lambda) = F_t \supseteq F_{t-1} \supseteq \cdots \supseteq F_0 = (0),$$

with $F_j/F_{j-1} \cong \Delta^{\lambda_j}$ *for some* $\lambda_j \in \Lambda_{n-1}$*, and similarly for induced modules.*

The sequence $(A_n)_{n\geq 0}$ *is* strongly coherent *if, in addition,*

$$\lambda_t < \lambda_{t-1} < \cdots < \lambda_1$$

in Λ_{n-1}*, and similarly for induced modules.*

Strong coherence and path bases

Let $(A_n)_{n\geq 0}$ be a strong coherent sequence of cellular algebras over R, and suppose the algebras A_n^F over the field of fractions F of R are (split) semisimple. Then the simple modules for A_n^F have bases indexed by paths on the branching diagram (Bratteli diagram) $\mathfrak{B}$ for the sequence $(A_k)_{0\leq k\leq n}$.

It is natural to want bases of the cell modules of A_n also indexed by paths on $\mathfrak{B}$ and having good properties with respect to restriction. Call such bases of the cell modules and their globalization to cellular bases of A_n *path bases*

With mild additional assumptions, satisfied in our examples, one always has *path bases* in strongly coherent towers.

Example of strong coherence

The sequence of Hecke algebras $H_n(q)$ is a strongly coherent sequence of cellular algebras, and the Murphy basis is a path basis.

This results from combining theorems of Murphy, Dipper-James, and Jost from the 80's

Main theorem

Theorem 4

Suppose $(A_n)_{n\geq 0}$ *and* $(Q_n)_{n\geq 0}$ *are two sequence of* $*$ *- algebras over* R*. Let* F *be the field of fractions of* R*. Assume:*

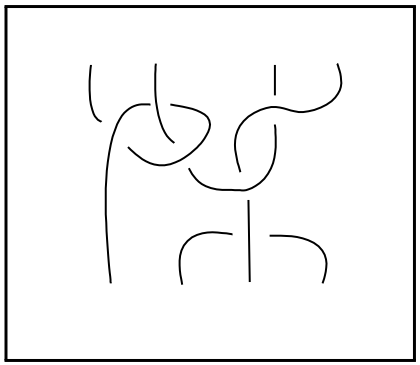
1. $(Q_n)_{n\geq 0}$ *is a (strongly) coherent tower of cellular algebras.*
2. $A_0 = Q_0 = R$, $A_1 \cong Q_1$.
3. *For each* $n \geq 2$, A_n *has an essential idempotent* $e_{n-1} = e_{n-1}^*$ *and* $A_n / A_n e_{n-1} A_n \cong Q_n$.
4. $A_n^F = A_n \otimes_R F$ *is split semisimple.*
5. (a) e_{n-1} commutes with A_{n-1}, and $e_{n-1} A_{n-1} e_{n-1} \subseteq A_{n-2} e_{n-1}$, (b) $A_n e_{n-1} = A_{n-1} e_{n-1}$ and $x \mapsto x e_{n-1}$ is injective from A_{n-1} to $A_{n-1} e_{n_1}$, and (c) $e_{n-1} = e_{n-1} e_n e_{n-1}$.

Then $(A_n)_{n\geq 0}$ *is a (strongly) coherent tower of cellular algebras.*

Example: The BMW algebras

A chief example is $A_n =$ BMW (Birman-Murakami-Wenzl) algebra on n strands, and $Q_n =$ Hecke algebra on n strands. BMW algebras arise in knot theory (Kauffman link invariant) and in quantum invariant theory (Schur–Weyl duality for quantum groups of symplectic and orthogonal types).

The BMW algebra is an algebra of braid like objects, namely

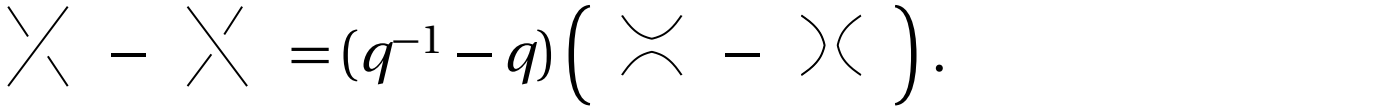
framed (n, n)*–tangles* :  .

Tangles can be represented by quasi-planar diagrams as shown here. Tangles are multiplied by stacking (like braids).

Definition of BMW algebras

Definition 5

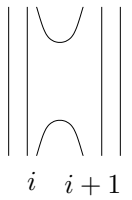
Let S be a commutative unital ring with invertible elements ρ, q, δ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$. The BMW algebra W_n^S is the S–algebra of framed (n,n)–tangles, modulo the Kauffman skein relations:

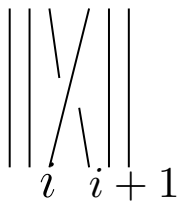
1. *(Crossing relation)*

2. *(Untwisting relation)*

3. *(Free loop relation)* $T \cup \bigcirc = \delta T$, *where* $T \cup \bigcirc$ *is the union of a tangle* T *and an additional closed loop with zero framing.*

The BMW algebras, cont.

- W_n^S imbeds in W_{n+1}^S. On the level of tangle diagrams, the embedding is by adding one strand on the right.
- The BMW algebras have an S–linear algebra involution, acting by turning tangle diagrams upside down.
- The following tangles generate the BMW algebra

$e_i =$ 

and $g_i =$ 

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The element e_i is an essential idempotent with $e_i^2 = \delta e_i$. One has $e_i e_{i\pm1} e_i = e_i$.

The BMW algebras, cont. 2

The ideal J_n generated by one or all e_i's in W_n^S satisfies $W_n^S/J_n \cong H_n^S(q^2)$ where $H_n^S(q^2)$ is the Hecke algebra.

There is a generic ground ring for the BMW algebras, namely

$$R = \mathbb{Z}[\boldsymbol{q}^{\pm 1}, \boldsymbol{\rho}^{\pm 1}, \boldsymbol{\delta}]/J,$$

where J is the ideal generated by

$$\boxed{\boldsymbol{\rho}^{-1} - \boldsymbol{\rho} - (\boldsymbol{q}^{-1} - \boldsymbol{q}^{-1})(\boldsymbol{\delta} - 1)}$$

and where the bold symbols denote indeterminants.

BMW – Generic ground ring, cont. 3

The generic ground ring R is an integral domain, with field of fractions $F = \mathbb{Q}(q, \rho)$, and $\delta = (\rho^{-1} - \rho)/(q^{-1} - q) + 1$ in F.

For every instance of the BMW over a ring S with parameters ρ, q, δ, one has $W_n^S \cong W_n^R \otimes_R S$.

The BMW algebras over F are semisimple (theorem of Wenzl).

The BMW algebras, application of our theorem

Now let's see what's involved in applying the theorem to the BMW algebras (with $A_n = W_n^R$, and $Q_n = H_n^R(q^2)$). Hypothesis (1) is the (strong) coherence of the sequence of Hecke algebras, which is a significant theorem about Hecke algebras. Hypothesis (4) on the semisimplicity of W_n^F is Wenzl's theorem. Everything else is elementary, and already contained in Birman-Wenzl.

All the other examples work pretty much the same way.

Some idea of the proof

The proof is inductive and is a cellular version of Wenzl's semisimplicity proof. Suppose we know that A_k is cellular (and satisfies all the conclusions of the theorem) for $k \leq n$.

Then we want to show the same for A_{n+1}. The main point is to show that $J_{n+1} = A_{n+1} e_n A_{n+1} = A_n e_n A_n$ is a "cellular ideal" in A_{n+1}. (This suffices to show cellularity, because we also have that $A_{n+1}/J_{n+1} \cong Q_{n+1}$ is cellular by hypothesis, and extensions of cellular algebras by cellular ideals are also cellular.)

Some idea of the proof–cont.

We have a Λ_{n-1}–cell net
$\Gamma \mapsto J(\Gamma) := \mathrm{span}\{c^{\lambda}_{\mathfrak{s},\mathfrak{t}} : \lambda \in \Gamma, \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\lambda)\}$. Now we want to show that

$$\Gamma \mapsto \hat{J}(\Gamma) := A_n e_n J(\Gamma) A_n$$

is a Λ_{n-1}–cell net in $A_n e_n A_n$. Along the way to doing this we also have to show that

$$J'(\Gamma) := A_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} A_n \cong A_n e_n J(\Gamma) A_n$$

and in particular $A_n \otimes_{A_{n-1}} A_n \cong A_n e_n A_n$, and if $\Gamma_1 \subseteq \Gamma_2$, then also $J'(\Gamma_1)$ imbeds in $J'(\Gamma_2)$. This is a bit tricky because $A_n e_n A_n$ is not a unital algebra and A_n is not projective as A_{n-1}–module.

Last slide!

Now if $\lambda \in \Lambda_{n-1}$ and $\Gamma_1 \subseteq \Gamma_2$, with $\Gamma_2 \setminus \Gamma_1 = \lambda$, then

$$\begin{aligned}
\hat{J}(\Gamma_2)/\hat{J}(\Gamma_1) &\cong J'(\Gamma_2)/J'(\Gamma_1) \\
&\cong A_n \otimes_{A_{n-1}} J(\Gamma_1)/J(\Gamma_2) \otimes_{A_{n-1}} A_n \\
&\cong A_n \otimes_{A_{n-1}} (\Delta^\lambda \otimes_R (\Delta^\lambda)^*) \otimes_{A_{n-1}} A_n \\
&\cong (A_n \otimes_{A_{n-1}} \Delta^\lambda) \otimes_R (\Delta^\lambda)^* \otimes_{A_{n-1}} A_n)
\end{aligned}$$

Now we need that $M^\lambda = A_n \otimes_{A_{n-1}} \Delta^\lambda$ is free as R–module, to verify the crucial property in the definition of a cell net. But as A_n–module, M^λ is $\mathrm{Ind}(\Delta^\lambda)$, and by an induction assumption on coherence of the cellular structures on $(A_k)_{k \le n}$, this has a filtration by cell modules for A_n, so is free as an R–module.

Last slide + 1

There are also some results about lifting Jucys–Murphy elements from Q_n to A_n in our setting, but I don't want to give details. The method recovers, with a very simple proof, some results of John Enyang (for Brauer and BMW algebras) and of Rui and Si in the cyclotomic Brauer and BMW cases.