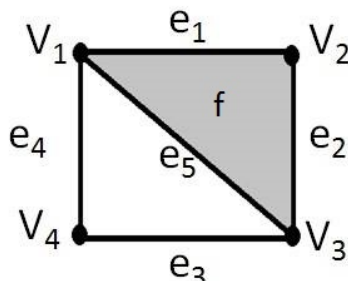# Homework 4: Solutions

Calculate the following for all i for the simplicial complex from class:

Figure 1: Simplicial complex from class.

1. Find C_i, B_i, Z_i.

   Note that C_i, B_i, Z_i and H_i are all vector spaces with $\mathbb{Z}_2$ coefficients. A vector space is determine by its basis. Thus one can find C_i, B_i, Z_i and H_i by finding bases for these vector spaces. The rank of a finite-dimensional vector space is the number of elements in a basis for that vector space.

   Recall the maps $\partial_i : C_i \to C_{i-1}$ are boundary operators.
   Recall also that ∂_i is a linear map: $\partial_i(\Sigma_j n_j \sigma_j) = n_j \Sigma_j \partial_i(\sigma_j)$ where $\Sigma_j n_j \sigma_j$ is an i-chain $=$ a linear combination of i-dimensional simplices. Thus to understand ∂_i, we just need to know how it acts on a simplex, σ_j.

   (a) C_i = set of i-dimensional chains = set of all linear combination of i-dimensional simplices = the vector space where the basis is the set of i-dimensional simplices.

   - $C_0 = \mathbb{Z}_2[v_1, v_2, v_3, v_4] =< v_1, v_2, v_3, v_4 >$

   - $C_1 = \mathbb{Z}_2[e_1, e_2, e_3, e_4, e_5] =< e_1, e_2, e_3, e_4, e_5 >$

   - $C_2 = \mathbb{Z}_2[f] =< f >$

   - $C_i = \{0\}$ for all $i > 2$ because there are no i-dimensional simplices in this simplicial complex for $i > 2$.

   (b) $Z_i = \{\Sigma_j n_j \sigma_j \text{ in } C_i \mid \partial_i(\Sigma_j n_j \sigma_j) = 0\}$

   - Determine Z_0.

     Method 1: $Z_0 = \{\Sigma_i n_i v_i \text{ in } C_0 \mid \partial_0(\Sigma_i n_i v_i) = 0\} =< v_0, v_1, v_2, v_3 >= C_0$ since ∂_0 takes all the vertices (and thus all 0-chains) to 0.

     Method 2: Using matrices. From question 2: $M_0 = \begin{matrix} v_1 & v_2 & v_3 & v_4 \\ \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$

     $Z_0 =$ null space of $M_0 =< v_1, v_2, v_3, v_4 >$

- Determine Z_1.

  Method 1: $Z_1 = \{\Sigma_i n_i e_i \text{ in } C_1 \mid \partial_1(\Sigma_i n_i e_i) = 0\}$.

  $\partial_1(\Sigma_i n_i \sigma_i) = 0$ implies $\partial_1(n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4 + n_5 e_5) = 0$.

  By linearity, $n_1 \partial_1(e_1) + n_2 \partial_1(e_2) + n_3 \partial_1(e_3) + n_4 \partial_1(e_4) + n_5 \partial_1(e_5) = 0$.

  Thus, $n_1(v_1 + v_2) + n_2(v_2 + v_3) + n_3(v_3 + v_4) + n_4(v_1 + v_4) + n_5(v_1 + v_3) = 0$

  Hence, $(n_1 + n_4 + n_5)v_1 + (n_1 + n_2)v_2 + (n_2 + n_3 + n_5)v_3 + (n_3 + n_4)v_4 = 0$.

  This implies $n_1 + n_4 + n_5 = 0$, $n_1 + n_2 = 0$, $n_2 + n_3 + n_5 = 0$, $n_3 + n_4 = 0$.

  Hence, $n_1 = n_2$, $n_3 = n_4$ and $n_5 = n_1 + n_4 = n_2 + n_3 = 0$.

  Thus $\Sigma_i n_i e_i \in Z_1$ iff $\Sigma_i n_i e_i = n_1 e_1 + n_1 e_2 + n_4 e_3 + n_4 e_4 + (n_1 + n_4)e_5$
  $$= n_1(e_1 + e_2 + e_5) + n_4(e_3 + e_4 + e_5)$$

  Thus a 1-chain is in Z_1 iff
  $\qquad$ it is a linear combination of the 1-chains $(e_1 + e_2 + e_5)$ and $(e_3 + e_4 + e_5)$.

  Thus $\{(e_1 + e_2 + e_5), (e_3 + e_4 + e_5)\}$ is a basis for Z_1.

  Thus, $Z_1 = \{\Sigma_i n_i e_i \text{ in } C_1 \mid \partial_1(\Sigma_i n_i e_i) = 0\}$
  $\quad = \{\Sigma_i n_i e_i \text{ in } C_1 \mid n_1 + n_4 + n_5 = 0, n_1 = n_2, n_3 = n_4\} = < e_1 + e_2 + e_5, e_3 + e_4 + e_5 >$

  Method 2: Using matrices. From question 2:

$$M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \overset{\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array}}{\left(\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)} \rightarrow \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \overset{\begin{array}{ccccc} e_1 & e_2 & e_3 & e_3+e_4+e_5 & e_5 \end{array}}{\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)}$$

$$\rightarrow \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \overset{\begin{array}{ccccc} e_1 & e_2 & e_3 & e_3+e_4+e_5 & e_1+e_2+e_5 \end{array}}{\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)}$$

  $Z_1 = $ null space of $M_1 = < e_1 + e_2 + e_5, e_3 + e_4 + e_5 >$

- Determine Z_2

  Method 1: $Z_2 = \{\Sigma_i n_i f_i$ in $C_2 \mid \partial_2(\Sigma_i n_i f_i) = 0\} = \{0\}$ since $\partial_2(f) = e_1 + e_2 + e_5 \neq 0$.

  Alternatively, note that $C_2 = \{0, f\}$. Since ∂_2 is a linear map, $\partial_2(0) = 0$. Thus $0 \in Z_2$. $\partial_2(f) = e_1 + e_2 + e_5 \neq 0$. Thus $f \notin Z_2$. Thus $Z_2 = $ kernel of $\partial_2 = \{x$ in $C_2 \mid \partial_2(x) = 0\} = \{0\}$

  Method 2: Using matrices. From question 2: $M_2 = \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \overset{f}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$

  $Z_2 = $ null space of $M_2 = \{0\}$.

- Determine Z_i for $i > 2$.

  $Z_i = \{0\}$ for all $i > 2$ since $C_i = \{0\}$ for all $i > 2$.

(c) Determine $B_i = $ image of ∂_{i+1}

- Determine B_0

  Method 1: $B_0 = $ image of $\partial_1 = \ <\ \partial(e_1),\ \ \partial(e_2),\ \ \partial(e_3),\ \ \partial(e_4),\ \ \partial(e_5)\ >$
  $= \ < v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1, v_1 + v_3 >.$

  Note, $v_1 + v_3 = (v_1 + v_2) + (v_2 + v_3)$.
  Thus the last generator of $< v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1, v_1 + v_3 >$ is a linear combination of the first two generators. Thus $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1, v_1 + v_3\}$ is NOT a linearly independent set.

  Also, $v_4 + v_1 = (v_1 + v_2) + (v_2 + v_3) + (v_3 + v_4)$

  Since the last two generators are linear combinations of the first 3 generators,

  $$< v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1, v_1 + v_3 > \ = \ < v_1 + v_2, v_2 + v_3, v_3 + v_4 >$$

  Note that $\{v_1 + v_2, v_2 + v_3, v_3 + v_4\}$ is a linearly independent set and thus this set forms a basis for B_0.

  Thus $B_0 \ = \ < v_1 + v_2, v_2 + v_3, v_3 + v_4 >.$

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Method 2: Using matrices. From question 2:

$$
M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array}
\left(\begin{array}{ccccc}
1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0
\end{array}\right)
\rightarrow
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{ccccc} e_1 & e_2 & e_3 & e_1+e_2+e_3+e_4 & e_5 \end{array}
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0
\end{array}\right)
$$

$$
\rightarrow
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{ccccc} e_1 & e_2 & e_3 & e_1+e_2+e_3+e_4 & e_1+e_2+e_5 \end{array}
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{array}\right)
$$

B_0 = the image of the column space of $M_1 = < v_1 + v_2, v_2 + v_3, v_3 + v_4 >$.

- Determine B_1

  Method 1: B_1 = image of $\partial_2 = \; < \partial_2(f) > \; = \; < e_1 + e_2 + e_5 >$

  Alternatively, note that $C_2 = \{0, f\}$. Since ∂_2 is a linear map, $\partial_2(0) = 0$. Thus $0 \in B_1$. $\partial_2(f) = e_1 + e_2 + e_5$. Thus $e_1 + e_2 + e_5 \in B_1$.

  Thus B_1 = image of $\partial_2 = \{0, e_1 + e_2 + e_5\} = < e_1 + e_2 + e_5 >$

  Method 2: Using matrices. From question 2: $M_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} f \end{array} \left(\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{array}\right)$

  B_1 = the image of the column space of $M_2 = < e_1 + e_2 + e_5 >$

- Determine B_i for $i \geq 2$

  B_2 = image of ∂_3 where $\partial_3 : C_3 \rightarrow C_2$. Since $C_3 = \{0\}$, $B_2 = \{0\}$.

  Similarly $B_i = \{0\}$ for $i > 2$. Thus $B_i = \{0\}$ for $i \geq 2$.

2. Find the matrix corresponding to each boundary map (from C_i to C_{i-1}).

(a) Let M_0 be the matrix corresponding to the boundary map $\partial_0 : C_0 \to 0$. Then, ∂_0 takes every vertex to 0.

$$
\text{Thus } M_0 = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \end{array} \right) \end{array}
$$

(b) Let M_1 be the matrix corresponding to the boundary map $\partial_1 : C_1 \to C_0$.
Since $\text{rank}(C_1) = 5$ and $\text{rank}(C_0) = 4$, M_1 is a 4×5 matrix. Also, a_{ij} is nonzero iff $a_{ij} = 1$ iff the vertex corresponding to row i is in the boundary of the edge corresponding to column j. For example, we have that $\partial(e_1) = v_1 + v_2$; hence we have a 1 as an entry in both the first and second rows of the first column. The remaining entries in the first column are zero.

$$
M_1 = \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \left(\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array} \right) \end{array}
$$

(c) Let M_2 be the matrix corresponding to the boundary map $\partial_2 : C_2 \to C_1$.
Since $\text{rank}(C_2) = 1$ and $\text{rank}(C_1) = 5$, M_2 is a 5×1 matrix. Since $\partial(f) = e_1 + e_2 + e_5$, we have a 1 as an entry in the rows corresponding to those edges.

$$
M_2 = \begin{array}{c} \begin{array}{c} f \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \left(\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{array} \right) \end{array}
$$

3. Find $H_i = \frac{Z_i}{B_i}$.

Recall that $\partial_{i+1}(\partial_i(c)) = 0$ for any c in C_{i+1}. Hence we get the inclusion $B_i \subset Z_i$. By modding out Z_i by B_i, we are taking all the elements in Z_i that are also in B_i and setting them equal to 0.

(a) $H_0 = Z_0/B_0 = \frac{<v_1, v_2, v_3, v_4>}{<v_1+v_2, v_2+v_3, v_3+v_4>}$

$$= <v_1, v_2, v_3, v_4 \mid v_1 + v_2 = 0, v_2 + v_3 = 0, v_3 + v_4 = 0> = <[v_1]>$$

where $[v_1] = \{v_1, v_2, v_3, v_4\}$ is a representative of the set containing all the vertices. Since we are working with coefficients in $\mathbb{Z}_2$, we get that $\{v_1 + v_2 = 0, v_2 + v_3 = 0, v_3 + v_4 = 0\}$ implies $v_1 = v_2$, $v_2 = v_3$, $v_3 = v_4$. Thus $[v_1] = \{v_1, v_2, v_3, v_4\}$

Rank H_0 = Rank Z_0 - Rank $B_0 = 4 - 3 = 1$

(b) $H_1 = Z_1/B_1 = \frac{<e_1+e_2+e_5, e_3+e_4+e_5>}{<e_1+e_2+e_5>}$

$$= <e_1 + e_2 + e_5, e_3 + e_4 + e_5 \mid e_1 + e_2 + e_5 = 0> = <[e_3 + e_4 + e_5]>$$

The cycle $e_1 + e_2 + e_5$ was filled by the face f and thus $e_1 + e_2 + e_5 = 0$ in H_1.

Rank H_1 = Rank Z_1 - Rank $B_1 = 2 - 1 = 1$

Sidenote: To determine H_1, we set every element in B_1 equal to 0.
Thus $\partial(f) = e_1 + e_2 + e_5 = 0$. Thus in H_1,
$e_3+e_4+e_5 = (e_3+e_4+e_5)+0 = (e_3+e_4+e_5)+\partial(f) = (e_3+e_4+e_5)+(e_1+e_2+e_5) = e_1+e_2+e_3+e_4$.

Thus $e_1+e_2+e_3+e_4 \in [e_3+e_4+e_5]$ in H_1. This also means that $[e_1+e_2+e_3+e_4] = [e_3+e_4+e_5]$ and thus $H_1 = <e_3 + e_4 + e_5> = <e_1 + e_2 + e_3 + e_4>$.

Hence $H_1 = <e_3 + e_4 + e_5>$ and $H_1 = <e_1 + e_2 + e_3 + e_4>$ are both correct answers.

(c) $H_2 = Z_2/B_2 = <0>$ since $Z_2 = <0>$.

(d) Similarly, $H_i = <0>$ for all $i > 2$.