22m:033 Notes: 1.7 Linear Independence

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1 Linear Independence

Definition 1.1 *A set of vectors* $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_p}\}$ *of* R^n *is* ***linearly dependent*** *if the vector equation*

$$x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + \cdots x_p\overrightarrow{v_p} = \overrightarrow{0}$$

has a nontrivial solution.

Such a solution is called a ***linear independence relation****.*

A set of vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_p}\}$ *of* R^n *is* ***linearly independent*** *if the vector equation*

$$x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + \cdots x_p\overrightarrow{v_p} = \overrightarrow{0}$$

has only the trivial solution.

Example 1.2 If we have

$$\overrightarrow{v_1} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \overrightarrow{v_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{v_3} = \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix}$$

then these three vectors are linearly dependent since

$$5\overrightarrow{v_1} + 3\overrightarrow{v_2} + 2\overrightarrow{v_3} = \overrightarrow{0}.$$

Note that by using a dependence equation like this we can solve for any of the vectors in terms of the others.

For example
$$\overrightarrow{v_1} = -\frac{3}{5}\overrightarrow{v_2} - \frac{2}{5}\overrightarrow{v_3}$$
and in this sense "$\overrightarrow{v_1}$ depends on $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$".

Remark 1.3 In general we note (Theorem 7 of text) if a set of vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_p}\}$ of R^n is linearly dependent if at least one of them is a linear combination of the others.

Remark 1.4 A single non zero vector constitutes a linearly independent set.

A set of two non-zero vectors is linearly independent if one is not a multiple of the other.

Remark 1.5 The columns of a matrix A are linearly independent if and only if the equation $A\overrightarrow{x} = \overrightarrow{0}$ has *only* the trivial solution since we can write $A\overrightarrow{x} = \overrightarrow{0}$ as
$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots x_p\overrightarrow{a_p} = \overrightarrow{0}$$

where the $\overrightarrow{a_i}$ are the column vectors.

Remark 1.6 If we have a set of homogeneous equations with more variables than equations, then there are non trivial solutions.

This means (Theorem 8 of text) if we have a set of (non zero) vectors in R^n and the number of vectors of in the set is larger than the dimension n of R^n then this must be a linearly dependent set of vectors.

So a set of four (non zero) vectors in R^3 must be linearly dependent.

Example 1.7 For what values of g, if any, will the following vectors be linearly independent:

$$\overrightarrow{v_1} = \begin{pmatrix} 1 \\ 2 \\ g \end{pmatrix}, \overrightarrow{v_2} = \begin{pmatrix} 1 \\ g \\ 2 \end{pmatrix}, \overrightarrow{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

.

So we want to know: for what values of $c_1, c_2,$ and c_3 will the equation $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + c_3\overrightarrow{v_3} = \overrightarrow{0}$ have only the trivial solution.

As we have seen this corresponds to a homogeneous system of linear equations in the variables $c_1, c_2,$ and c_3:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 + gc_2 &= 0 \\ gc_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

Since this is a homogeneous system, we need only look at the coefficient matrix rather than the augmented matrix (although of course, we could used the augmented matrix if we wanted). This matrix and its row reduction is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & g & 0 \\ g & 2 & 1 \end{pmatrix}$$

Multiply the first row by -2 and add to row 2:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & g-2 & -2 \\ g & 2 & 1 \end{pmatrix}$$

Multiply the first row by $-g$ and add to row 3:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & g-2 & -2 \\ 0 & 2-g & 1-g \end{pmatrix}$$

Add row 2 to row 3:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & g-2 & -2 \\ 0 & 0 & -1-g \end{pmatrix}$$

We do not really need to complete to a reduced row echelon form to answer the question.

If both $g-2$ and $-1-g$ are non zero then the system will have a unique solution namely the trivial solution in which case our vectors are linearly independent. If $g=-1$ then the last row would be zero, and if $g=2$ the third row would be 0 0 -2, the third row would be 0 0 1 and we could reduce the matrix to have a row of zeros.

So we conclude: the vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$ are linearly independent for all values of g except $g=-1$ and $g=2$.