22m:033 Notes:
1.8 Linear Transformations

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1 Transformation

Definition 1.1 *A function $T\colon R^n \to R^m$ is sometimes called a* ***transformation*** *or a* ***mapping****.*

The set R^n is called the ***domain*** *of the transformation.*

If $\overrightarrow{x} \in R^n$, the vector $T(\overrightarrow{x})$ in R^m is called the ***image*** *of $\overrightarrow{x}$. The set of all images $T(\overrightarrow{x})$ is called the* ***range*** *of T.*

Note: many other textbooks call the range of T, ***the image of*** *T. We will sometimes use this in class.* ∎

2 Using a matrix to define a transformation

We can use the multiplication of a matrix and a vector to define a transformation: If A is an $n \times m$ matrix and $\overrightarrow{x} \in R^n$ then $A\overrightarrow{x}$ will be a vector in R^m. So we can define $T\colon R^n \to R^m$ by:

$$T(\overrightarrow{x}) = A\overrightarrow{x}$$

Example 2.1 If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ then the transformation $T(\overrightarrow{x}) = A\overrightarrow{x}$ is a mapping from R^2 to R^3.

For example

$$T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}$$

Question: is the vector $\overrightarrow{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in the range of T? In other words, can we solve $A\overrightarrow{x} = \overrightarrow{b}$ for some $\overrightarrow{x}$?

If we row reduce

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{pmatrix}$$

we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This means the equations have no solution so we conclude that $\overrightarrow{b}$ is not in the range of the transformation.

Note: it is important to keep in mind that the matrix we have considered is the augmented matrix for a set of equations and not just a coefficient matrix. ∎

3 Linear Transformation

Definition 3.1 *A transformation* $T\colon R^n \to R^m$ *is called a linear transformation if*

$$T(\overrightarrow{u} + \overrightarrow{v}) = T(\overrightarrow{u}) + T(\overrightarrow{v}) \text{ for all } \overrightarrow{u}, \overrightarrow{v} \in R^n$$
$$T(c\overrightarrow{u}) = cT(\overrightarrow{u}) \text{ for all } \overrightarrow{u} \in R^n, c \in R \ \blacksquare$$

Remark 3.2 Two important properties of a linear transformation are:

$$T(\overrightarrow{0}) = \overrightarrow{0}$$
$$T(c\overrightarrow{u} + d\overrightarrow{v}) = cT(\overrightarrow{u}) + dT(\overrightarrow{v}) \ \blacksquare$$

The next remark is important. It says that the composition of linear transformations is a linear transformation.

Remark 3.3 If $T\colon R^n \to R^m$ is a linear transformation and $S\colon R^m \to R^k$,is a linear transformation, then the composite $T \circ S\colon R^n \to R^k$ is a linear transformation.

Proof:

$$(T\circ S)(\overrightarrow{u}+\overrightarrow{v}) = T(S(\overrightarrow{u}+\overrightarrow{v})) \text{ definition of composition}$$
$$T(S(\overrightarrow{u}) + S(\overrightarrow{v})) \ S \text{ is linear}$$
$$T(S(\overrightarrow{u})) + T(S(\overrightarrow{v}))) \ T \text{ is linear}$$
$$(T \circ S)(\overrightarrow{u}) + (T \circ S)(\overrightarrow{v})) \text{ definition of composition}$$

Similarly

$$(T \circ S)(c\overrightarrow{u}) = c(T \circ S)(\overrightarrow{u}) \ \blacksquare$$

4 Optional—calculus and linear transformations

NOTE: this section will not be covered on any homework problems or tests. I mention to give an idea of how profound some of our concepts in basic understanding of mathematics.

Fundamental concepts of calculus are linear transformations.

In the first place we can consider the set real valued functions F of a single variable as "vectors"— since we can add them and multiply them by numbers and the numbers R is a vector space.

Basically we can say:

Proposition 4.1 *The derivative is a linear transformation of F to F and the definite integral gives a linear transformation from F to R.*

5 Matrix Transformations and linear transformations

A transformation defined using a matrix is a linear transformation.

In a section we will skip it is shown that any linear transformation $T\colon R^n \to R^m$ is a matrix transformation for some matrix.

6 The geometry of linear transformations

A linear transformation sends straight lines to straight lines or to a point.

Recall a line in vector parametric form is $\overrightarrow{y} = t\overrightarrow{m} + \overrightarrow{b}$. and so

$$T(\overrightarrow{y}) = T(t\overrightarrow{m} + \overrightarrow{b}) = tT(\overrightarrow{m}) + T(\overrightarrow{b}),$$

which is a vector equation of a line.

Example 6.1 Consider the transformation $T(\overrightarrow{x}) = A\overrightarrow{x}$ with matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

The x- axis is sent to the diagonal line $\overrightarrow{y} = \overrightarrow{x}$ since

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}.$$

On the other hand all points on the line $\overrightarrow{y} = -\overrightarrow{x}$ are sent to a point, namely $\overrightarrow{0}$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ -a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad \blacksquare$$

Question 6.2 For the transformation of Example 6.1 there is a line L so that all points of L are mapped by T to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find a vector equation for L.

For linear transformations of the plane it is useful to see what happens to the triangle with vertices $(0,0), (1,0),$ and $(1,1)$. See other class documents for some examples.

7 The meaning of the columns of a matrix considered as a linear transformation

In the plane we have special vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Geometrically these are unit vectors along axes. Similarly in three dimensions we have:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In two dimensions, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $T(\overrightarrow{x}) = A\overrightarrow{x}$, then

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} b \\ d \end{pmatrix}$$

So the column vectors of a matrix are the images of these standard unit vectors. It is easy to see that this holds for matrices of other sizes—not just square matrices.

For example for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ Then $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, etc.

8 Finding a matrix for a transformation

Is there a linear transformation T given by a matrix A so that

$$T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

If we write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This gives us four linear equations in four unknowns

$a, b, c,$ and d:

$$\begin{aligned} a + 2b &= 2 \\ c + 2d &= 1 \\ a + b &= -1 \\ c + d &= 1 \end{aligned}$$

These clearly separate into two sets of two equations with two unknowns which we can solve.

Question 8.1 What is A in the above example.