22m:033 Notes:
5.2 The Characteristic Equation

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/∼roseman

April 15, 2010

1 Finding eigenvalues: an example

Example 1.1 Let us try to find the eigenvalues (if any) for $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$. So we are looking for values λ such that

$$(A - \lambda I)\overrightarrow{x} = \overrightarrow{0}$$

has non trivial solutions and this will happen if and only if the $\det(A - \lambda I) = 0$

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) && (1) \\ &= \det\left(\begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}\right) && (2) \\ &= (3-\lambda)(-1-\lambda) && (3) \end{aligned}$$

So we must have either $\lambda = -1$ or $\lambda = 3$.

Now lets see what eigenvectors correspond to $\lambda = 3$.

We look for the eigenspace of $A - 3I = \begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}$.

This is the same as the null space of $\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}$ which is the same as the nulspace of $\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$. Now as before we view $\overrightarrow{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. We have one free variable, expressed say as $y = t$. then $2x - t = 0$ or $x = \frac{t}{2}$ and we see the null space is all vectors of the form

$$\begin{pmatrix} \frac{t}{2} \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$

So the eigenspace associated with $\lambda = 3$ is one dimensional with basis $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$

Lets check this:

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 3 \end{pmatrix} = 3 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

So we see that $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ is, indeed, an eigenvector with eigenvalue 3.

Next we look at the eigenvalue $\lambda = -1$. We need to look at the null space of

$$A - (-1)I = A + I = \begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}$$

This matrix row reduces to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Letting $y = t$ we see that the eigenvectors corresponding to this eigenvalue are the vectors $t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2 More generally...

Definition 2.1 *Given a square matrix A the* ***characteristic polynomial of*** A ***with variable*** λ *is*

$$\det(A - \lambda I).$$

What we have discussed may be summed up as follows:

Proposition 2.2 *A number λ is an eigenvalue for matrix A if and only if it is a zero of the characteristic polynomial of A.*

Remark 2.3 I have "good news" and I have "bad news".

The "good news" is that by the above result all we have to do to find eigenvalues is to find zeros of a polynomial.

The "bad news" is that finding *exact* zeros is not a simple matter. This can be hard if the polynomial is a cubic, difficult if it is a degree 4 polynomial and "not possible by any one formula" if it is a polynomial of degree 5 or more.

Example 2.4 Lets try and find eigenvalues for $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$. Then the characteristic polynomial is

$$\det\left(\begin{pmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{pmatrix}\right) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

So all we have to do is to find the roots of $\lambda^3 - 6\lambda^2 + 11\lambda - 6$. It is probably time to crack open the old algebra book.

3 Algebra review: roots of polynomials

For some relatively simple polynomials here is a result that can be very helpful.

Proposition 3.1 *Suppose* $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ *is a polynomial with all of the* a_i *integers. If a rational number* $\frac{p}{q}$ *is a zero then* q *divides* a_n *evenly and* p *divides* a_0 *evenly.*

Also " if r is a root then $(x - r)$ is a factor":

Proposition 3.2 *If* $p(x)$ *is a polynomial with zero* r *then we can write* $p(x) = (x - r)p_1(x)$ *where* $p_1(x)$ *is a polynomial.*

Example 3.3 Consider the polynomial of Example 2.4, $\lambda^3 - 6\lambda^2 + 11\lambda - 6$. If this has a rational zero $\frac{p}{q}$ then q must divide $a_3 = 1$ evenly and so $q = \pm 1$. Also p must divide 6. So the possible zeros are: $\pm 1, \pm 2, \pm 3, \pm 6$.

Lets try -1. We see that -1 is *not* a zero since

$$(-1)^3 - 6(-1)^2 + 11(-1) - 6 = -1 - 6 - 11 - 6 \neq 0.$$

We next try +1 and find that +1 *is* a zero since

$$(+1)^3 - 6(+1)^2 + 11(+1) - 6 = 1 - 6 + 11 - 6 = 0.$$

So $(\lambda - 1)$ must be a factor of our polynomial. We can find our p_1 by long division of polynomials.

When we do we find that

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda^2 - 5\lambda + 6)$$

recalling how to factor quadratics we now see that

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

We conclude that our matrix has three eigenvalues: 1, 2 and 3. ∎

Remark 3.4 We note that finding eigenvalues for triangular matrices is very easy. If A is triangular then $A - \lambda I$ is also triangular and we have seen that the determinant of a triangular matrix is the product of the diagonal entries. This provides us a characteristic polynomial already completely factored.

Example 3.5 If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$ when we calculate $\det(A - \lambda I)$ we see that

$$\det\left(\begin{pmatrix} 1-\lambda & 2 & 3 & 4 \\ 0 & 5-\lambda & 6 & 7 \\ 0 & 0 & 8-\lambda & 9 \\ 0 & 0 & 0 & 10-\lambda \end{pmatrix}\right) = (1-\lambda)(5-\lambda)(8-\lambda)(10-\lambda$$

And so our matrix has eigenvalues 1, 5, 8 and 10.

4 Similarity

The following definition is a natural one if one thinks about linear transformations associated with a square matrix. We will see this later. At the moment it just appears in the text without motivation.

Definition 4.1 *Suppose A and B are $n \times n$ matrices. We say A is similar to B if there is an $n \times n$ invertible matrix P such that*

$$B = P^{-1}AP.$$

Remark 4.2 It is not hard to show that if A is similar to B then B is similar to A.

If $B = P^{-1}AP$ then $A = PBP^{-1}$. If we write $Q = P^{-1}$ then Q is invertible, $Q^{-1} = (P^{-1})^{-1} = P$ and so we can write $B = Q^{-1}AQ$ showing that B is similar to A.

An important property of similar matrices is:

Proposition 4.3 *If A and B are similar matrices, then they have the same characteristic polynomial.*

The proof of this uses some matrix algebra and properties of determinants: Suppose B is similar to A. we calculate the characteristic polynominal of B as:

$$det(B - \lambda I) = det(P^{-1}AP - \lambda(P^{-1}P)) =$$

$$det\left((P^{-1}(A - \lambda I)P)\right) = det(P^{-1})det(A - \lambda I)det(P) =$$

$$(det(P^{-1})det(P))det(A - \lambda I) = (1)det(A - \lambda I) = det(A - \lambda I)$$

5 Problems

1. Factor $x^3 - 8x^2 + 17x - 4$
2. Factor $x^4 + x^3 - 7x^2 - x + 6$ Hint: $x = -3$ is one root.
3. Write the long division calculation needed in Example 2.4 that shows:

$$\frac{\lambda^3 - 6\lambda^2 + 11\lambda - 6}{\lambda - 1} = \lambda^2 - 5\lambda + 6.$$

4. In Example 2.4 find a basis for the eigenspace corresponding to $\lambda = 2$.