## 1.5, 1.6 Characterizations of Invertible Matrices

### The Invertible Matrix Theorem

Let $A$ be a square $n \times n$ matrix. The the following statements are equivalent (i.e., for a given $A$, they are either all true or all false).

a. $A$ is an invertible matrix.

b. $A$ is row equivalent to $I_n$.

c. $A$ has $n$ pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. $A$ is expressible as a product of elementary matrices.

f. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

g. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

h. There is an $n \times n$ matrix $C$ such that $CA = I_n$.

i. There is an $n \times n$ matrix $D$ such that $AD = I_n$.

j. $A^T$ is an invertible matrix.

**EXAMPLE:** Use the Invertible Matrix Theorem to determine if $A$ is invertible, where

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix}.$$

*Solution*

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix} \sim \cdots \sim \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 16 \end{bmatrix}}_{\text{3 pivots positions}}$$

*Circle correct conclusion:* Matrix $A$ *is* / *is not* invertible.

**Theorem**

Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.

**EXAMPLE:** Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$?

If not, find all $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

**Solution:** Augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ is ______ consistent for all $\mathbf{b}$ since some choices of $\mathbf{b}$ make $-2b_1 + b_3$ nonzero.

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if

$$-2b_1 + b_3 = 0.$$

(equation of a plane in $\mathbf{R}^3$)