Introduction to Determinants

Recall that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det [a] = a$.

The **minor of** a_{ij} is the determinant of the matrix obtained from matrix A by deleting the ith row and jth column of A. Notation: M_{ij}.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad M_{23} = \det \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = \left(a_{ij} \right)$ is given by

$$\det A =$$
$$a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{1+n} a_{1n}M1n$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j}M_{1j}$$

EXAMPLE: Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution

$$\det A =$$

$$1 \det \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \underline{\hspace{6cm}} = \underline{\hspace{2cm}}$$

Common notation: $\det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}.$

So

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

The $(\mathbf{i,j})$-**cofactor** of A is the number C_{ij} where $C_{ij} = (-1)^{i+j} M_{ij}$.

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

THEOREM 1   The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
(expansion across row i)

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$
(expansion down column j)

EXAMPLE: Compute the determinant of
$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ using cofactor expansion down column 3.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} =$$

$$0 \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

EXAMPLE: Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} + 0 \begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 3 & 5 \end{vmatrix} -$$

$$0 \begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 2 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 14$$

Method of cofactor expansion is not practical for large matrices -

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$
$$\text{(upper triangular)} \qquad \text{(lower triangular)}$$

THEOREM 2: If A is a triangular matrix, then det A is the product of the main diagonal entries of A.

EXAMPLE:

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \underline{\hspace{4cm}} = -24$$

Properties of Determinants

THEOREM 3   Let A be a square matrix.

a.   If a multiple of one row of A is added to another row of A to produce a matrix B, then $\det A = \det B$.

b.   If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c.   If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Theorem 3(c) indicates that $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} =$

$k \begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}$.

EXAMPLE: Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix}$$

$$= 5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \underline{\hspace{2cm}} =.$$

EXAMPLE: Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

EXAMPLE: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.

Solution

$$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2 \, (1) \, (-1) \, (-6) = -12.$$

COROLLARY   If A is an $n \times n$ matrix with two proportional rows (columns) then

$$\det A = 0.$$

THEOREM 4   If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Partial proof $(2 \times 2$ case)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \text{and}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

$$\Rightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

$(3 \times 3$ case)

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = a \begin{vmatrix} e & h \\ f & i \end{vmatrix} - b \begin{vmatrix} d & g \\ f & i \end{vmatrix} + c \begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$\Rightarrow \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

Implications of Theorem 4?

Theorem 3 still holds if the word *row* is replaced

with _____.

Suppose A has been reduced to

$$U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$$ by row replacements

and row interchanges, then

$$\det A =$$
$$\begin{cases} (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

THEOREM 5   A square matrix is invertible if and only if $\det A \neq 0$.

THEOREM 6 (Multiplicative Property)

For $n \times n$ matrices A and B,

$$\det(AB) = (\det A)(\det B).$$

EXAMPLE: Compute $\det A^3$ if $\det A = 5$.

Solution:

$$\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$$

$$= \underline{} = \underline{}.$$

EXAMPLE: For $n \times n$ matrices A and B, show that A is singular if $\det B \neq 0$ and $\det AB = 0$.

Solution: *Since*

$$(\det A)(\det B) = \det AB = 0$$

and

$$\det B \neq 0,$$

then $\det A = 0$. Therefore A is singular.