

## 4.1 Vector Spaces & Subspaces

Many concepts concerning vectors in  $\mathbf{R}^n$  can be extended to other mathematical systems.

We can think of a *vector space* in general, as a collection of objects that behave as vectors do in  $\mathbf{R}^n$ . The objects of such a set are called *vectors*.

A **vector space** is a nonempty set  $V$  of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  in  $V$  and for all scalars  $c$  and  $d$ .

1.  $\mathbf{u} + \mathbf{v}$  is in  $V$ .
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
3.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector)  $\mathbf{0}$  in  $V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$ .
5. For each  $\mathbf{u}$  in  $V$ , there is vector  $-\mathbf{u}$  in  $V$  satisfying  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
6.  $c\mathbf{u}$  is in  $V$ .
7.  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ .
8.  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ .
9.  $(cd)\mathbf{u} = c(d\mathbf{u})$ .
10.  $1\mathbf{u} = \mathbf{u}$ .

## Vector Space Examples

**EXAMPLE:** Let  $M_{2 \times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ are real} \right\}$

In this context, note that the  $\mathbf{0}$  vector is  $\begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix}$ .

**EXAMPLE:** Let  $n \geq 0$  be an integer and let

$\mathbf{P}_n =$  the set of all polynomials of degree at most  $n \geq 0$ .

Members of  $\mathbf{P}_n$  have the form

$$\mathbf{p}(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$$

where  $a_0, a_1, \dots, a_n$  are real numbers and  $t$  is a real variable. The set  $\mathbf{P}_n$  is a vector space.

*We will just verify 3 out of the 10 axioms here.*

Let  $\mathbf{p}(t) = a_0 + a_1t + \cdots + a_nt^n$  and  $\mathbf{q}(t) = b_0 + b_1t + \cdots + b_nt^n$ . Let  $c$  be a scalar.

**Axiom 1:**

The polynomial  $\mathbf{p} + \mathbf{q}$  is defined as follows:

$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$ . Therefore,

$$\begin{aligned}(\mathbf{p} + \mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (\text{_____}) + (\text{_____})t + \cdots + (\text{_____})t^n\end{aligned}$$

which is also a \_\_\_\_\_ of degree at most

\_\_\_\_\_. So  $\mathbf{p} + \mathbf{q}$  is in  $\mathbf{P}_n$ .

**Axiom 4:**

$$\begin{aligned}\mathbf{0} &= 0 + 0t + \cdots + 0t^n \\ &\text{(zero vector in } \mathbf{P}_n\text{)}\end{aligned}$$

$$\begin{aligned}(\mathbf{p} + \mathbf{0})(t) &= \mathbf{p}(t) + \mathbf{0} = (a_0 + 0) + (a_1 + 0)t + \cdots + (a_n + 0)t^n \\ &= a_0 + a_1t + \cdots + a_nt^n = \mathbf{p}(t)\end{aligned}$$

and so  $\mathbf{p} + \mathbf{0} = \mathbf{p}$

**Axiom 6:**

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (\text{_____}) + (\text{_____})t + \cdots + (\text{_____})t^n$$

which is in  $\mathbf{P}_n$ .

The other 7 axioms also hold, so  $\mathbf{P}_n$  is a vector space.

## Subspaces

Vector spaces may be formed from subsets of other vector spaces. These are called *subspaces*.

A **subspace** of a vector space  $V$  is a subset  $H$  of  $V$  that has three properties:

- a. The zero vector of  $V$  is in  $H$ .
- b. For each  $\mathbf{u}$  and  $\mathbf{v}$  in  $H$ ,  $\mathbf{u} + \mathbf{v}$  is in  $H$ . (In this case we say  $H$  is closed under vector addition.)
- c. For each  $\mathbf{u}$  in  $H$  and each scalar  $c$ ,  $c\mathbf{u}$  is in  $H$ . (In this case we say  $H$  is closed under scalar multiplication.)

*If the subset  $H$  satisfies these three properties, then  $H$  itself is a vector space.*

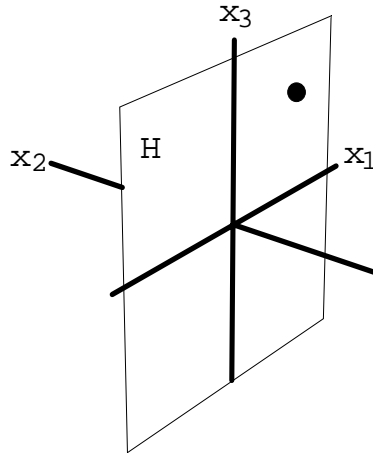
**EXAMPLE:** Let  $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$ . Show that  $H$  is a subspace of  $\mathbf{R}^3$ .

*Solution:* Verify properties a, b and c of the definition of a subspace.

- The zero vector of  $\mathbf{R}^3$  is in  $H$  (let  $a = \underline{\hspace{2cm}}$  and  $b = \underline{\hspace{2cm}}$ ).
- Adding two vectors in  $H$  always produces another vector whose second entry is  $\underline{\hspace{2cm}}$  and therefore the sum of two vectors in  $H$  is also in  $H$ . ( $H$  is closed under addition)
- Multiplying a vector in  $H$  by a scalar produces another vector in  $H$  ( $H$  is closed under scalar multiplication).

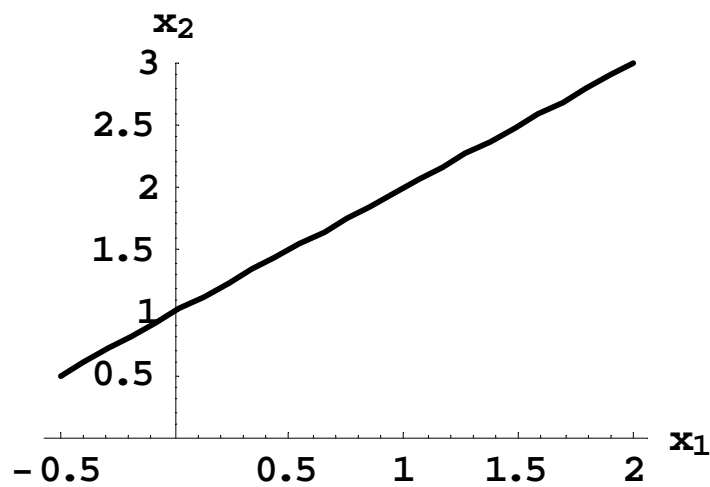
Since properties a, b, and c hold,  $V$  is a subspace of  $\mathbf{R}^3$ .

**Note:** Vectors  $(a, 0, b)$  in  $H$  look and act like the points  $(a, b)$  in  $\mathbb{R}^2$ .



**EXAMPLE:** Is  $H = \left\{ \begin{bmatrix} x \\ x + 1 \end{bmatrix} : x \text{ is real} \right\}$  a subspace of \_\_\_\_\_?

I.e., does  $H$  satisfy properties a, b and c?



*Graphical Depiction of H*

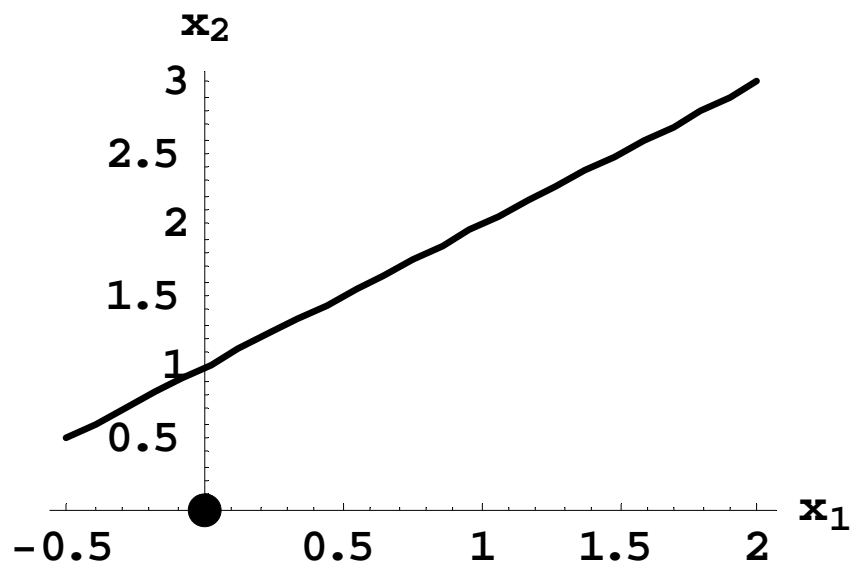
*Solution:*

*All three properties must hold in order for  $H$  to be a subspace of  $\mathbf{R}^2$ .*

Property (a) is not true because

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Therefore  $H$  is not a subspace of  $\mathbf{R}^2$ .



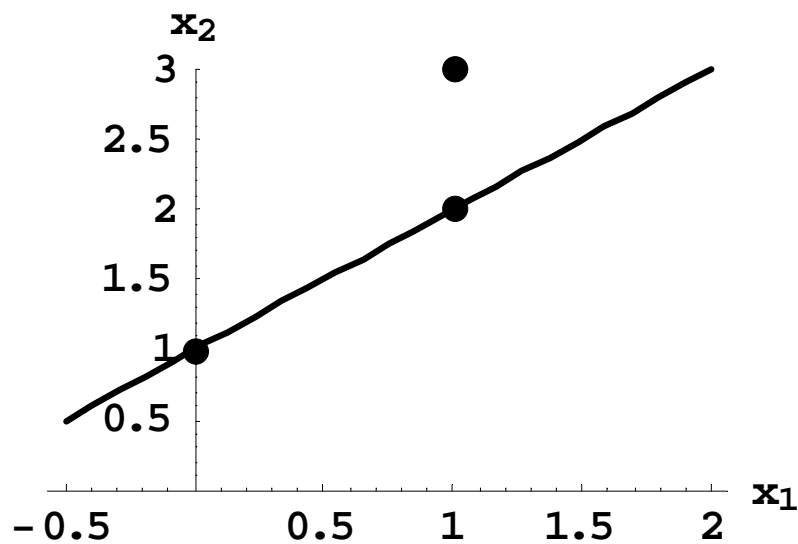
Property (a) fails

Another way to show that  $H$  is not a subspace of  $\mathbf{R}^2$ :

Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ then } \mathbf{u} + \mathbf{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

and so  $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ , which is \_\_\_\_\_ in  $H$ . So property (b) fails and so  $H$  is not a subspace of  $\mathbf{R}^2$ .



Property (b) fails



## A Shortcut for Determining Subspaces

### THEOREM 1

If  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are in a vector space  $V$ , then  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a subspace of  $V$ .

**Proof:** In order to verify this, check properties a, b and c of definition of a subspace.

a.  $\mathbf{0}$  is in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  since

$$\mathbf{0} = \underline{\hspace{1cm}} \mathbf{v}_1 + \underline{\hspace{1cm}} \mathbf{v}_2 + \cdots + \underline{\hspace{1cm}} \mathbf{v}_p$$

b. To show that  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  closed under vector addition, we choose two arbitrary vectors in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  :

$$\mathbf{u} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_p \mathbf{v}_p$$

and

$$\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p.$$

Then

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_p \mathbf{v}_p) + (b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p) \\ &= (\underline{\hspace{1cm}} \mathbf{v}_1 + \underline{\hspace{1cm}} \mathbf{v}_1) + (\underline{\hspace{1cm}} \mathbf{v}_2 + \underline{\hspace{1cm}} \mathbf{v}_2) + \cdots + (\underline{\hspace{1cm}} \mathbf{v}_p + \underline{\hspace{1cm}} \mathbf{v}_p) \\ &= (a_1 + b_1) \mathbf{v}_1 + (a_2 + b_2) \mathbf{v}_2 + \cdots + (a_p + b_p) \mathbf{v}_p. \end{aligned}$$

So  $\mathbf{u} + \mathbf{v}$  is in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ .

c. To show that  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  closed under scalar multiplication, choose an arbitrary number  $c$  and an arbitrary vector in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  :

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$\begin{aligned} c\mathbf{v} &= c(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p) \\ &= \text{_____}\mathbf{v}_1 + \text{_____}\mathbf{v}_2 + \cdots + \text{_____}\mathbf{v}_p \end{aligned}$$

So  $c\mathbf{v}$  is in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ .

Since properties a, b and c hold,  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a subspace of  $V$ .

## Recap

1. To show that  $H$  is a subspace of a vector space, use Theorem 1.
2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms a, b or c (from the definition of a subspace) is violated.

**EXAMPLE:** Is  $V = \{(a + 2b, 2a - 3b) : a \text{ and } b \text{ are real}\}$  a subspace of  $\mathbf{R}^2$ ? Why or why not?

*Solution:* Write vectors in  $V$  in column form:

$$\begin{aligned} \begin{bmatrix} a + 2b \\ 2a - 3b \end{bmatrix} &= \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} 2b \\ -3b \end{bmatrix} \\ &= \text{---} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \text{---} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \end{aligned}$$

So  $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$  and therefore  $V$  is a subspace of \_\_\_\_\_ by Theorem 1.

**EXAMPLE:** Is  $H = \left\{ \begin{bmatrix} a + 2b \\ a + 1 \\ a \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$  a

subspace of  $\mathbf{R}^3$ ? Why or why not?

*Solution:*  $\mathbf{0}$  is not in  $H$  since  $a = b = 0$  or any other combination of values for  $a$  and  $b$  does not produce the zero vector. So property \_\_\_\_\_ fails to hold and therefore  $H$  is not a subspace of  $\mathbf{R}^3$ .

**EXAMPLE:** Is the set  $H$  of all matrices of the form

$\begin{bmatrix} 2a & b \\ 3a + b & 3b \end{bmatrix}$  a subspace of  $M_{2 \times 2}$ ? Explain.

*Solution:* Since

$$\begin{bmatrix} 2a & b \\ 3a + b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$$

$$= a \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}.$$

Therefore  $H = \text{Span} \left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}$  and so  $H$  is a subspace of  $M_{2 \times 2}$ .