22M027: Introduction to Linear Algebra

- Linear equations.
- Geometrical interpretation.
- Number of solutions of a linear system.
- Solving linear systems by row reductions.
- Row-echelon form and reduced row-echelon form of a matrix.
- Matrix operations (addition, scalar mutiplication, multiplication including block multiplication) and their properties; matrix transpose.
- Matrix inverse: elementary matrices, finding matrix inverse, using inverses in solving linear syste;
- Invertible Matrix Theorem.
- Scalar, diagonal, triangular and symmetric matrices and their properties.
- Determinants: evaluating by cofactor expansion.
- Determinants: properties; evaluating by row reduction.
- Application of Determinants: Cramer's rule, inverse of a Matrix.
- Vector operation, norm, dot product.
- Projections, distance between a point and a line.
- Cross product: properties, applications (area of a parallelogram, volume of a parallelepiped).
- Equations of a plane (normal vector and a point; three points.
- Parametric equations of a line.
- When planes intersect, are parallel, perpendicular.
- Distance between a point and a plane, two parallel planes.

Review Problems

1. Solve the following system of linear equations

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= -1 \\ 2x_1 + x_2 - 2x_3 - 2x_4 &= -2 \\ -x_1 + 2x_2 - 4x_3 + x_4 &= 1 \\ 3x_1 \qquad\qquad\quad - 3x_4 &= -3 \end{aligned}$$

2. Use block product to find the product AB of the following matrices

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \qquad B := \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. A is a 2-by-5 matrix and B is a 3-by-2.

   (a) Is the product AB defind? If yes, state its size.

   (b) Is the product BA defind? If yes, state its size.

4. Find the product CD of the following matrices

$$C := \begin{bmatrix} 1 & 2 \\ -5 & 0 \\ 2 & -3 \end{bmatrix}, \qquad D := \begin{bmatrix} -1 & 0 & 1 & -2 \\ 10 & 2 & 5 & 3 \end{bmatrix}.$$

5. Find the inverse of the following matrix:

$$A := \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}.$$

   (a) using row reduction;

   (b) using adjoint matrix.

6. Solve the following linear system:

$$\begin{aligned} 2x_1 \qquad\quad + 3x_3 &= -3 \\ 3x_2 + 2x_3 &= 3 \\ -2x_1 \qquad\quad - 4x_3 &= 6 \end{aligned}$$

   (a) using row reduction algorithm;

   (b) using the inverse of the coefficient matrix;

   (c) using Cramer's rule.

7. Consider the following matrix

$$E := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & -1 & 7 & 4 \\ 1 & -2 & 5 & 9 \end{bmatrix}.$$

   (a) evaluate the determinant of E;

   (b) is this matrix E invertible? If yes, give the determinant of its inverse E^1.

8. Find the components of the vector having initial point $P_1(-1, 0, 2)$ and terminal point $P_2(0, -1, 0)$.

9. Find the distance between the points P_1 and P_2 above.

10. Find the components of the orthogonal projection of $\vec{u} = (-1, -2)$ on $\vec{a} = (-2, 3)$.

11. Find the angle between vectors $\vec{u} = (-1, -2)$ and $\vec{a} = (-2, 3)$.

12. Find a vector perpendicular to the line $2x + 3y - 5 = 0$.

13. Find the distance between the origin $O = (0, 0)$ and the line $2x + 3y - 5 = 0$.

14. Find the components of a vector perpendicular to both vectors $\vec{u} = (1, 2, 0)$ and $\vec{v} = (-2, 3, 0)$.

15. Find the area of a parallelogram defined by vectors $\vec{u} = (1, 2)$ and $\vec{v} = (-2, 3)$.

16. Find the volume of a parallelepiped defined by vectors $\vec{u} = (1, 2, 0)$, $\vec{v} = (-2, 3, 0)$, and $\vec{v} = (1, 1, 1)$.

17. Find an equation of the line passing through the points $P_1(-1, 0, 2)$ and $P_2(0, -1, 0)$.

18. Find an equation of the line passing through the point $P_1(0, -1, 0)$ and parallel to vector $(-1, 1, 2)$.

19. Find an equation of the plane passing through the point $P_1(0, -1, 0)$ and perpendicular to vector $(-1, 1, 2)$.

20. Find an equation of the plane passing through the points $P_1(-1, 0, 2)$, $P_2(0, -1, 0)$ and $P_3(1, 1, 1)$.

21. Are the two planes $2x + 3y - 5z = 10$ and $6x + 9y - 15z = 12$ parallel? Explain.

22. Are the two planes $2x + 3y - 2z = 10$ and $5x - 2y + 2z = 15$ perpendicular? Explain.

23. Find the distance between the origin $O = (0, 0, 0)$ and the plane $2x + 3y - 5z = 10$.

24. Find the distance between parallel planes $2x + 3y - 5z = 10$ and $6x + 9y - 15z = 12$.