Binomial coefficients

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Pascal's Formula

For integers n and k such that $1 \leq k \leq n$,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

$$\binom{n}{k} = 0 \quad \text{if} \quad k > n;$$

$$\binom{n}{0} = 1 \quad \text{for any} \quad n.$$

For integers n and k such that $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Theorem 1 (Pascal's Formula.) For integers n and k such that $1 \leq k \leq n-1$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The numbers in the second column of Pascal's triangle ($k = 2$): $\binom{n}{2} = \frac{n(n-1)}{2}$ are triangle numbers.
The numbers in the third column of Pascal's triangle ($k = 3$): $\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$ are tetrahedral numbers.

The Binomial Theorem

Theorem 2 (Binomial Expansion.) For integer $n \geq 1$ and variables x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$