Derangements

October 29, 2008

A permutation $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ is called a *derangement* if $i_k \neq k$ for any $1 \leq k \leq n$ (no number remains in its position).

Theorem 1. For $n \geq 1$, the number D_n of derangements of $\{1, 2, \ldots, n\}$ is given by

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}).$$

Corollary 2. $\lim_{n \to \infty} \frac{D_n}{n!} = e^{-1}$.

Here are a few derangement numbers:

$$D_1 = 0, \quad D_2 = 1, \quad D_3 = 2, \quad D_4 = 9, \quad D_5 = 44, \quad D_6 = 265.$$

Proposition 3. The derangement sequence D_n satisfies the recurrence relation

$$D_n = (n-1)(D_{n-1} + D_{n-2}), \qquad n \geq 3$$

with the initial condition $D_1 = 0, D_2 = 1$.

Proposition 4. The derangement sequence D_n satisfies the recurrence relation

$$D_n = n D_{n-1} + (-1)^n, \qquad n \geq 2$$

with the initial condition $D_1 = 0$.

Problems

1. How many ways can a hatcheck girl hand back the n hats of n gentlemen, one to each gentleman, with no man gettnig his hat?

2. Determine the number of permutations of $\{1, 2, \ldots, n\}$ in which no even integer is in its natural position.

3. Determine the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k numbers displaced.

Nonconsecutive permutations

A permutation of $\{1, 2, \ldots, n\}$ is called *nonconsecutive* if none of $12, 23, \ldots, (n-1)n$ occurs.

Theorem 5. For $n \geq 1$, the number Q_n of nonconsecutive permutations of $\{1, 2, \ldots, n\}$ is given by

$$Q_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!$$

Proposition 6. For $n \geq 2$,

$$Q_n = D_n + D_{n-1}.$$

A circular permutation of $\{1, 2, \ldots, n\}$ is called *nonconsecutive* if none of $12, 23, \ldots, (n-1)n, n1$ occurs.

Theorem 7. For $n \geq 1$, the number C_n of nonconsecutive circular permutations of $\{1, 2, \ldots, n\}$ is given by

$$C_n = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k-1)! + (-1)^n$$

Problems

1. n camels march in the desert in a caravan. Their journey is very uneventful, and the camels are tired of always watching the same animals in front of them. Therefore their owner wants to rearrange them so that every camel has a different camel in front of it (the position of the original first camel is arbitrary). How many possible rearrangements are there?

2. How many ways are there for 8 children on a merry-go-round to change places so that somebody new is in front of each child? (The horses are indistinguishable and in a circle.)

Surjective Functions

Let X be a set with m objects and let Y be a set with n objects.

The number of all functions from X to Y equals n^m.

The number of injective functions from X to Y equals $\binom{n}{m} m! = P(n, m)$.

Theorem 8. The number $S(m, n)$ of surjective functions from X to Y is given by

$$S(m, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

$$\sum_{i_1 + i_2 + \cdots + i_n = m, i_1, i_2, \ldots, i_n \geq 1} \binom{m}{i_1, i_2, \ldots, i_n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$