# Exponential Generating Functions

November 14, 2008

The *exponential generating function* of an infinite sequence

$$a_0, a_1, a_2, \ldots, a_k, \ldots$$

is the infinite series

$$A^{(e)}(x) = a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \cdots + a_k\frac{x^k}{k!} + \ldots$$

Let

$$\begin{aligned} A^{(e)}(x) &= a_0 + b_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \cdots + a_k\frac{x^k}{k!} + \ldots \\ B^{(e)}(x) &= b_0 + b_1\frac{x}{1!} + b_2\frac{x^2}{2!} + \cdots + b_k\frac{x^k}{k!} + \ldots \end{aligned}$$

Then

$$A^{(e)}(x)B^{(e)}(x) = \sum_{k=0}^{\infty}(\sum_{i=0}^{k}\binom{k}{i}a_i b_{k-i})\frac{x^k}{k!}$$

| $a_i$ | $1$ | $c^i$ | $i$ | $i^2$ | $i!$ | $(n)_i$ | $n^{(i)}$ |
|---|---|---|---|---|---|---|---|
| $A^{(e)}(x)$ | $e^x$ | $e^{cx}$ | $xe^x$ | $x(x+1)e^x$ | $(1-x)^{-1}$ | $(1+x)^n$ | $(1-x)^{-n}$ |

**Theorem 1.** Let $M = \{m_1 \cdot x_1, m_2 \cdot x_2, \ldots, m_n \cdot x_n\}$ be a multiset. Let $(a_k)$ be the number of permutations of the multiset $M$. Then the exponential generating function of the sequence $(a_k, k \geq 0)$ is given by

$$\prod_{i=1}^{n}(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{m_i}}{m_i!}).$$

1. Find the number of $k$-permutations of the multiset $\{\infty \cdot x_1, \infty \cdot x_2, \ldots, \infty \cdot x_n\}$.
2. Find the number of $k$-permutations of the multiset $\{\infty \cdot x_1, \infty \cdot x_2, \ldots, \infty \cdot x_n\}$ that contain at least one element of each type.
3. Determine the number of ways to color the squares of a 1-by-n chessboard using the colors, red, white, and blue, if an even number of squares are colored red.