Generating Combinations

September 24, 2008

Let S be an n-set. For convenience of generating combinations, we take S to be the set

$$S = \{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}.$$

Each subset A of S can be identified as a function $\chi_A : S \to \{0, 1\}$, called the characteristic function of A, defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

In practice, χ_A is represented by a $0-1$ sequence or a base 2 numeral. For example, for $S = \{x_7, x_6, x_5, x_4, x_3, x_2, x_1, x_0\}$,

$\emptyset$	00000000	0
$\{x_7, x_5, x_2, x_1\}$	10100110	
$\{x_6, x_5, x_3, x_1, x_0\}$	01101011	
$\{x_7, x_6, x_5, x_4, x_3, x_2, x_1, x_0\}$	11111111	

Algorithm for generating combinations of $\{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}$:
Begin with $a_{n-1}a_{n-2}\cdots a_1a_0 = 00\ldots00$.
While $a_{n-1}a_{n-2}\cdots a_1a_0 \neq 11\ldots11$, do
(1) Find the smallest integer j such that $a_j = 0$.
(2) Replace a_j by 1 and each of $a_{j-1}, \ldots, a_1, a_0$ by 0.
The algorithm stops when $a_{n-1}a_{n-2}\cdots a_1a_0 = 11\ldots11$.
For $n = 3$, the algorithm produces the list

000	100
001	101
010	110
011	111.

For $n = 4$, the algorithm produces the list

0000	0100	1000	1100
0001	0101	1001	1101
0010	0110	1010	1110
0011	0111	1011	1111.

1. For $n = 8$, find the position of the combination $\{x_6, x_5, x_3, x_1, x_0\}$ in the list.

2. Which combination comes after $\{x_6, x_5, x_3, x_1, x_0\}$?

Gray code order

The *unit n-cube* Q_n is a graph whose vertex set is the set of all $0-1$ sequences of length n, and two sequences are adjacent if they differ in only one place.
A *Gray code of order* n is a path of Q_n that visits every vertex of Q_n exactly once, i.e., a Hamilton path of Q_n. For example,

$$000 \longrightarrow 001 \longrightarrow 101 \longrightarrow 100 \longrightarrow 110 \longrightarrow 010 \longrightarrow 011 \longrightarrow 111$$

is a Gray code of order 3. It is obvious that this Gray code can not be a part of any Hamilton cycle since 000 and 111 are not adjacent.
A cyclic Gray code of order n is a Hamilton cycle of Q_n. For example, the closed path

$$000 \longrightarrow 001 \longrightarrow 011 \longrightarrow 010 \longrightarrow 110 \longrightarrow 111 \longrightarrow 101 \longrightarrow 100 \longrightarrow 000$$

is a cyclic Gray code of order 3.

Reflected Gray codes (inductive definition).

For $n = 1$, we have the Gray code $0 \longrightarrow 1$.
For $n = 2$, we use $0 \longrightarrow 1$ to produce the path $00 \longrightarrow 01$ by adding a 0 in the front, and use $1 \longrightarrow 0$ to produce $11 \longrightarrow 10$ by adding a 1 in the front, then combine the two paths to produce the Gray code $00 \longrightarrow 01 \longrightarrow 11 \longrightarrow 10$.
For $n = k$, we combine the Gray code of order $k-1$ with 0 added in the front of each $(k-1)$-tuple, and the Gray code of order $k-1$ in the reverse order with 1 added in the front of each $(k-1)$-tuple.
The Gray codes obtained in this way are called *reflected Gray codes*.

Reflected Gray codes (algorithm).

Algorithm for generating reflected Gray codes of order n:
Begin with $a_{n-1}a_{n-2}\cdots a_1a_0 = 00\ldots00$.
While $a_{n-1}a_{n-2}\cdots a_1a_0 \neq 10\ldots00$, do
(1) If $a_{n-1}+a_{n-2}+\cdots+a_1+a_0 = 00\ldots00 =$ even, then change a_0 (from 0 to 1 or 1 to 0).
(2) If $a_{n-1}+a_{n-2}+\cdots+a_1+a_0 = 00\ldots00 =$ odd, find the smallest j such that $a_j = 1$ and change a_{j+1} (from 0 to 1 or 1 to 0).

1. Construct the reflected Gray code of order $n = 3$ using the algorithm.
2. For $n = 8$, which 8-tuples follow 10100110 and 00011111 in the reflected Gray code?