Generating Functions

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The (ordinary) *generating function* of an infinite sequence

$$a_0, a_1, a_2, \dots, a_n, \dots$$

is the infinite series

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \dots$$

A finite sequence $a_0, a_1, a_2, \dots, a_n$ can be regarded as the infinite sequence

$$a_0, a_1, a_2, \dots, a_n, 0, 0, \dots$$

and its generating function $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is a polynomial.

sequence	generating function
$1, 1, 1, 1, 1, \dots$	$\frac{1}{1-x}$
$1, -1, 1, -1, 1, \dots$	$\frac{1}{1+x}$
$1, 0, 1, 0, 1, \dots$	$\frac{1}{1-x^2}$
$0, 1, 2, 3, 4, 5, \dots$	$\frac{1}{x(1-x)^{-2}}$
$1, 2, 4, 8, 16, \dots$	$\frac{1}{1-2x}$
$\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$	$(1+x)^n$
$\binom{\alpha}{0}, \binom{\alpha}{1}, \dots, \binom{\alpha}{i}, \dots$	$(1+x)^\alpha$
$\binom{n-1}{0}, \binom{n}{1}, \binom{n+1}{2} \dots, \binom{n+i-1}{i}, \dots$	$(1-x)^{-n}$
$0, 1, \frac{1}{2}, \frac{1}{3}, \dots$	$\ln \frac{1}{1-x}$

Problems

1. Find the number of ways to collect \$15 from 20 distinct people if each of the first 19 people can give a dollar (or nothing) and the 20th person can give either \$1 or \$5 (or nothing).

2. How many ways are there to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls but any amount can go unto each of the other six boxes.