Operations on Generating Function

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Let

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \dots \\ B(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \dots \end{aligned}$$

Then

$$\begin{aligned} cA(x) + dB(x) &= \sum_{i=0}^{\infty} (ca_i + db_i) x^i \\ A(cx) &= a_0 + (ca_1)x + (c^2 a_2)x^2 + \cdots + (c^n a_n)x^n + \dots \\ x^m A(x) &= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots + a_n x^{m+n} + \dots \\ \frac{A(x) - a_0 - a_1 x - \cdots - a_{m-1} x^{m-1}}{x^m} &= a_m + a_{m+1} x + a_{m+2} x^2 + \cdots + a_{m+n} x^n + \dots \\ A'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1) a_{n+1} x^n + \dots \\ xA'(x) &= 0a_0 + 1a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots + n a_n x^n + \dots \\ A(x)B(x) &= \sum_{i=0}^{\infty} (\sum_{i=0}^{k} a_i b_{k-i}) x^k \end{aligned}$$

1. Find the closed formula for the generating function of the sequence of m 1-s.
2. Solve the recurrence relation $a_n = 4a_{n-2}, a_0 = 0, a_1 = 1$.
3. Solve the system of recurrence relations

$$\begin{cases} b_n = 3^{n-1} - c_{n-1} \\ c_n = 3^{n-1} - b_{n-1} \end{cases}$$

   $b_0 = c_0 = 0$.

4. Evaluate the sum $\sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}$.
5. Evaluate the sum $\sum_{i=0}^{n-1} \binom{i+m-1}{m-1}$.