Generating r-Combinations

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Let S be an n-set

$$S = \{1, 2, \ldots, n-1, n\}$$

with a natural order

$$1 < 2 < 3 < \cdots < n.$$

For simplicity, we write an r-combination $\{a_1, a2, \ldots a_r\}$ as an r-permutation

$$a_1 a_2 \cdots a_r \quad \text{with} \quad a_1 < a_2 < \cdots < a_r.$$

For two r-combinations $A = a_1 a_2 \cdots a_r$ and $B = b_1 b_2 \cdots b_r$ of S, we say that A *precedes B in the lexicographic order*, written $A < B$, if there is an integer k $(1 \leq k \leq r)$ such that

$$a_1 = b_1, a_2 = b_2, \ldots, a_{k-1} = b_{k-1}, a_k < b_k.$$

The first r-combination of S in lexicographic order is $12 \cdots r$, and the last r-combination in lexicographic order is $(n-r+1) \cdots (n-1)n$.

Theorem. Let $a_1 a_2 \cdots a_r$ be an r-combination of $S = \{1, 2, \ldots, n-1, n\}$. If $a_1 a_2 \cdots a_r \neq (n-r+1) \cdots (n-1)n$ and k is the largest integer such that $a_k < n$ and $a_k + 1$ is not in the $a_1 a_2 \cdots a_r$, then the successor of $a_1 a_2 \cdots a_r$ is

$$a_1 a_2 \cdots a_{k-1}(a_k+1)(a_k+2) \cdots (a_k+r-k+1).$$

Algorithm for generating r-combinations of $S = \{1, 2, \ldots, n-1, n\}$:
Begin with $12 \cdots r$.
While $a_1 a_2 \cdots a_r \neq (n-r+1) \cdots (n-1)n$, do
(1) Find the largest integer k such that $a_k < n$ and $a_k + 1$ is not in the $a_1 a_2 \cdots a_r$.
(2) Replace $a_1 a_2 \cdots a_r$ with

$$a_1 a_2 \cdots a_{k-1}(a_k+1)(a_k+2) \cdots (a_k+r-k+1).$$

The collection of all 4-combinations of $\{1, 2, 3, 4, 5, 6\}$ are listed by the algorithm:

1234	1245	1345	1456	2356
1235	1246	1346	2345	2456
1236	1256	1356	2346	3456.

Theorem. Let r-combination $a_1a_2\cdots a_r$ of $S=\{1,2,\ldots,n-1,n\}$ occurs in place number

$$\binom{n}{r}-\binom{n-a_1}{r}-\binom{n-a_2}{r-1}-\cdots-\binom{n-a_{r-1}}{2}-\binom{n-a_r}{1}$$

in the lexicographic order.

1. Find the position of the 4-combination 1258 in the list of all 4-combinations of $\{1,2,3,4,5,6,7,8\}$.
2. Generate all 3-combinations of $\{1,2,3,4,5\}$.
3. Generate all 3-permutations of $\{1,2,3,4,5\}$.