Combinations of Multisets

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Let M be a multiset. **An r-combination** of M is an unordered collection of r objects of M. For a multiset $M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$ an r-combination of M is also called an r-combination *with repetition allowed* of the n-set $S = \{a_1, a_2, \ldots, a_k\}$.

Theorem 1 The number of nonnegative integer solutions for the equation

$$x_1 + x_2 + \cdots + x_k = r$$

equals

$$\binom{k+r-1}{r} = \binom{k+r-1}{k-1}.$$

Theorem 2 The number of positive integer solutions for the equation

$$x_1 + x_2 + \cdots + x_k = r$$

equals

$$\binom{r-1}{k-1}.$$

Theorem 3 Let $M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$ be a multiset of k different types where each type has infinitely many elements. Then the number of r-combinations of the multiset M (the number of r-combinations with repetition allowed) equals

$$\binom{k+r-1}{r} = \binom{k+r-1}{k-1}.$$

Corollary 4 The number of ways to place r identical balls into k distinct boxes equals

$$\binom{k+r-1}{r} = \binom{k+r-1}{k-1}.$$

Corollary 5 The number of ways to place r identical balls into k distinct boxes such that no box remains empty equals

$$\binom{r-1}{k-1}.$$

1. Find the number of integer solutions for the equation

$$x_1 + x_2 + x_3 + x_4 = 10,$$

where $x_1 \geq 3, x_2 \geq 0, x_3 \geq -2, x_4 \geq 5$.

2. Find the number of nonnegative integer solutions for the equation

$$x_1 + x_2 + x_3 + x_4 < 19.$$

3. A bakery sells 8 different kinds of doughnuts. If the bakery has virtually unlimited supply of each kind, how many different options for a dozen of doughnuts are there? What if a box is to contain at least one of each kind of doughnuts?

4. In how many ways can 12 indistiguishable apples and 1 orange be distributed among three children in such a way that each child gets at least one piece of fruit?

5. Find the number of nondecreasing sequences of length r whose terms are taken from the set $\{1, 2, \ldots, k\}$.