The Multinomial Theorem

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Pascal's Formula

Multinomial coefficient:

$$\binom{n}{n_1, n_2, \ldots, n_t} = \frac{n!}{n_1! n_2! \cdots n_t!},$$

where $n_1 + n_2 + \cdots + n_t = n$.
Binomial coefficients are a particular case of multinomial coefficients:

$$\binom{n}{k} = \binom{n}{k, n-k}$$

Theorem 1 (Pascal's Formula for multinomial coefficients.) For integers $n, n_1, n_2, \ldots, n_t$ such that $n_1 + n_2 + \cdots + n_t = n$,

$$\binom{n}{n_1, n_2, \ldots, n_t} = \binom{n-1}{n_1 - 1, n_2, \ldots, n_t} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_t} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_t - 1}.$$

The Multinomial Theorem

Theorem 2 (Multinomial Expansion.) For integer $n \geq 1$ and variables $x_1, x_2, \ldots, x_k$,

$$(x_1 + x_2 + \cdots + x_t)^n = \sum_{n_1 + n_2 + \cdots + n_t = n n_1, n_2, \ldots, n_t \geq 0} \binom{n}{n_1, n_2, \ldots, n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.$$

1. Write down the expansion of $(x_1 + x_2 + x_3)^3$.
2. Find the coefficient of $x_1^2 x_3 x_4^3 x_5$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^7$.
3. Find the coefficient of $x_1^3 x_2 x_3^2$ in the expansion of $(2x_1 - 3x_2 + 5x_3)^6$.
4. How many different terms are there in the expansion of $(x_1 + x_2 + \cdots + x_t)^n$?
5. Prove that

$$\sum_{n_1 + n_2 + \cdots + n_t = n n_1, n_2, \ldots, n_t \geq 0} \binom{n}{n_1, n_2, \ldots, n_t} = t^n.$$