Pigeonhole Principle (Strong form)

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Let $q_1, q_2, \dots q_n$ be positive integers. If $q_1 + q_2 + \cdots + q_n - n + 1$ objects are put into n boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects, …,or the nth box contains at least q_n objects.

The strong form of the pigeonhole principle is most often applied in the special case when $q_1 = q_2 = \cdots = q_n = r$. In this case the principle becomes:

If $n(r-1)+1$ objects are put into n boxes, then at least one of the boxes contains r or more of the objects.

The simple form of the pigeonhole principle is obtained from the last staement by taking $r = 2$.

If the average of n nonnegative integers $a_1, a_2, \dots a_n$ is greater than $r-1$, i.e.,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} > r - 1,$$

then at least one of the integers is greater than or equal to r.

1. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?

2. There are 30 classes and 1000 students in a school. Prove that at least one class has at least 34 students.

3. Given two disks, one smaller than the other. Each disk is divided into 200 congruent sectors. In the larger disk 100 sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The smaller disk is placed on the larger disk so that the centers and sectors coincide. Show that it is possible to align the two disks so that the number of sectors of the smaller disk whose color matches the corresponding sector of the larger disk is at least 100.

4. Show that every sequence $a_1, a_2, \dots, a_{n^2+1}$ of $n_2 + 1$ real numbers contains either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

5. 51 points are placed, in a random way, into a square of side 1 unit. Can we prove that 3 of these points can be covered by a circle of radius $\frac{1}{7}$ units ?