Partially Ordered Sets

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Definition. A relation on a set X is a subset R of the product set $X \times X$. A relation R on X is called

1. *reflexive* if xRx for all $x \in X$;
2. *irreflexive* if $x \not R x$ for all $x \in X$;
3. *symmetric* provided that if xRy for some $x, y \in X$ then yRx;
4. *antisymmetric* provided that if xRy and yRx for some $x, y \in X$ then $x = y$;
5. *transitive* provided that if xRy and yRz for some $x, y, z \in X$ then xRz.

Example.
(1) The relation of subset, $\subseteq$, is a reflexive and transitive relation on the power set $P(X)$.
(2) The relation of divisibility, $|$, is a reflexive and transitive relation on the set of positive integers.
A *partial order* on a set X is a reflexive, antisymmetric, and transitive relation.
A *strict partial order* on a set X is an irreflexive, antisymmetric, and transitive relation.
If a relation R is a partial order, we usually denote R by $\leq$; then the relation $<$ defined by $a < b$ if and only if $a \leq b$ but $a \neq b$ is a strict partial order.
Conversely, for a strict partial order $<$ on a set X, the relation $\leq$ defined by $a \leq b$ if and only if $a < b$ or $a = b$ is a partial order.
A set X with a partial order $\leq$ is called a *partially ordered set* (or *poset for short*) and is denoted by $(X, \leq)$.
Posets can be represented geometrically by *diagramms*. The *cover relation* $<_c$ is defined by

$$a <_c b \quad \text{iff} \quad a < b \quad \text{ and there is no } c \text{ such that } a < c < b.$$

1. Find the cover relation for $\subseteq$.
2. Find the cover relation for the relation of divisibility, $|$.
3. Draw the diagram representing the poset $(P(\{1, 2, 3\}), \subseteq)$.
4. Draw the diagram representing the poset $(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}), |)$.

A *linear order,* or *total order* on a set X is a strict order $<$ such that for any two distinct elements a and b, either $a < b$ or $b < a$.
An element a of a poset X is called *minimal* if $b \leq a$ implies $a = b$ for any $b \in X$.
An element a of a poset X is called the *smallest* if $a \leq b$ for any $b \in X$.
Theorem. Let X be a finite set. There is one-to-one correspondence between the total orders on X and the permutations of X.
Let $\leq_1$ and $\leq_2$ be two partial orders on a set X. The poset $(X, \leq_2)$ is called an *extension* of the poset $(X, \leq_1)$ if, whenever $a \leq_1 b$, then $a \leq_2 b$. In particular, an extension of a partial order has more comparable pairs.
We show that every finite poset has a *linear extension*, that is, an extension which is a linearly ordered set.

Theorem Let $(X, \leq)$ be a finite partially ordered set. Then there is a linear order $\preceq$ such that $(X, \preceq)$ is an extension of $(X, \leq)$.
Proof. We need to show that the elements of X can be listed in some order $\{x_1, x_2, \ldots, x_n\}$ in such a way that if $x_i \leq x_j$ then x_i comes before x_j in this list, i.e., $i \leq j$. The following algorithm does the job.

Algorithm for a linear extension of an n-poset:
Step 1. Choose a minimal element x_1 from X (with respect to the ordering $\leq$).
Step 2. Delete x_1 from X; choose a minimal element x_2 from X.
Step 3. Delete x_2 from X and choose a minimal element x_3 from X.
...
Step n. Delete x_{n-1} from X and choose the only element x_n in X.

1. Find a linear extention of $(\{1, 2, 3, \ldots, n\}, |)$.
2. Find a linear extention of $(P(\{1, 2, 3\}), \subseteq)$.

Equivalence Relations

A relation R on X is called an *equivalence relation* if R is reflexive, symmetric, and transitive. For an equivalence relation R on a set X and an element $x \in X$, we call the set $[x] = \{y \in X : xRy\}$ an *equivalence class* of R and x a *representative* of the equivalence class $[x]$.

A collection $P = \{A_1, A_2, \ldots A_k\}$ of nonempty subsets of a set X is called a *partition* of X if $A_i \bigcap A_j = \emptyset$ for $i \neq j$ and $X = \bigcup A_i$.
Theorem If R is an equivalence relation on a set X, then the collection

$$P_R = \{[x] : x \in X\}$$

is a partition of X.
Theorem If $P = \{A_1, A_2, \ldots A_k\}$ is a partition of X, then the relation

$$R_P = \bigcup A_i \times A_i$$

is an equivalence relation on X.