Linear Homogeneous Recurrence Relations

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A number sequence (x_n) is said to satisfy a *linear recurrence relation of order k* if

$$h_n = a_1(n)h_{n-1} + a_2(n)h_{n-2} + \cdots + a_k(n)h_{n-k} + b_n(n); \quad a_k(n) \neq 0, n \geq k \tag{1}$$

where the coefficients $a_1(n), a2(n), \ldots, a_k(n)$, and $b_n(n)$ are functions of n or constants. The linear recurrence relation (1) is called *homogeneous* if $b_n = 0$, and is said to have *constant coeffcients* if $a_1(n), a2(n), \ldots, a_k(n)$ are constants. The recurrence relation

$$h_n = a_1(n)h_{n-1} + a_2(n)h_{n-2} + \cdots + a_k(n)h_{n-k}; \quad a_k(n) \neq 0, n \geq k \tag{2}$$

is called *the corresponding homogeneous* linear recurrence relation of (1).

In this section we consider linear homogeneous recurrence relations of order k with constant coefficients of the form

$$h_n = a_1h_{n-1} + a_2h_{n-2} + \cdots + a_kh_{n-k}; \quad a_k \neq 0, n \geq k \tag{3}$$

where $a_1, a2, \ldots, a_k$ are constants. The following polynomial equation

$$x_k - a_1x^{k-1} - a_2x^{k-2} - \cdots - a_{k-1}x - a_0 = 0 \tag{4}$$

is called *the characteristic equation* associated with the recurrence relation (3).

Theorem 1. Let $q \neq 0$. The geometric sequence $h_n = q^n$ is a solution of the recurrence relation (3) if and only if the number q is a root of the characteristic equation (4).

Theorem 2. If the characteristic equation (4) has k distinct roots $q_1, q_2, \ldots, q_k$, then the general solution of (3) is

$$h_n = c_1q_1^n + c_2q_2^n + \cdots + c_kq_k^n.$$

Theorem 3. (a) Let q be a root with multiplicity m of the characteristic equation (4) associated with the recurrence relation (3). Then the m sequences $q^n, nq^n, n^2q^n, \ldots, n^{m-1}q^n$ are linearly independent solutions of the recurrence relation (3).

(b) Let $q_1, q_2, \ldots, q_s$ be distinct roots with the multiplicities $m_1, m_2, \ldots, m_s$ respectively for the characteristic equation (4). Then the sequences

$$\begin{array}{l} q_1^n, nq_1^n, n^2q_1^n, \ldots, n^{m_1-1}q_1^n; \\ q_2^n, nq_2^n, n^2q_2^n, \ldots, n^{m_2-1}q_2^n; \\ \cdots \\ q_s^n, nq_s^n, n^2q_s^n, \ldots, n^{m_s-1}q_s^n; \end{array}$$

are linearly independent solutions of the recurrence relation (3). Their linear combinations form the general solution of the recurrence relation (3).

Problems:

1. Find the general term of the Fibonacci sequence.
2. Find the sequence (h_n) satisfying the recurrence relation

$$h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}, \quad n \geq 3$$

   and the initial conditions $h_0 = 1, h_1 = 2$, and $h_2 = 0$.

3. Find the sequence (h_n) satisfying the recurrence relation

$$h_n = 4h_{n-1} - 4h_{n-2}, \quad n \geq 2$$

   and the initial conditions $h_0 = a$ and $h_1 = b$.

4. Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored using the colors red and blue so that no two squares that are colored red are adjacent. Find a recurrence relation that h_n satifies; then find a formula for h_n.