Number Sequences

October 31, 2008

An infinite number sequence is an ordered array

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

of countably many real or complex numbers, and is usually abbreviated as (a_n). A sequence (a_n) can be viewed as a function f from the set of nonnegative integers to the set of real or complex numbers, i.e., $f(n) = a_n, n = 0, 1, 2, \ldots$.

We call a sequence (a_n) an *arithmetic sequence* if it is of the form

$$a_0, a_0 + q, a_0 + 2q, \ldots, a_0 + nq, \ldots$$

The general term satisfies the recurrence relation

$$a_n = a_{n-1} + q, \quad n \geq 1.$$

A sequence (b_n) is called a *geometric sequence* if it is of the form

$$b_0, b_0 q, b_0 q^2, \ldots, b_0 q^n, \ldots$$

The general term satisfies the recurrence relation

$$b_n = b_{n-1} q, \quad n \geq 1.$$

The *partial sums* of a sequence (a_n) are the sums:

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_0 + a_1 \\ s_2 &= a_0 + a_1 + a_2 \\ &\cdots \end{aligned}$$

The partial sums form a new sequence (s_n).

For an arithmetic sequence $a_n = a_0 + nq \quad (n \geq 0)$, we have the partial sum

$$s_n = \sum_{k=0}^{n} (a_0 + kq) = (n+1)a_0 + \frac{qn(n+1)}{2}.$$

For a geometric sequence $b_n = b_0 q^n \quad (n \geq 0)$, we have

$$s_n = \sum_{k=0}^{n} (b_0 q^k) = \begin{cases} \frac{q^{n+1}-1}{q-1} b_0 & \text{if} \quad q \neq 1 \\ (n+1) b_0 & \text{if} \quad q = 1. \end{cases}.$$

The sequence $f_0, f_1, f_2, f_3, \ldots$ satisfying the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \quad (n \geq 2)$$

with the initial condition $f_0 = 0$ and $f_1 = 1$ is called *the Fibonacci sequence*, and the terms in the sequence are called *Fibonacci numbers*.

Proposition 1. The partial sum of Fibonacci sequence is $s_n = f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

Theorem 2. The general term of the Fibonacci sequence (f_n) is given by

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n, \quad (n \geq 0)$$