Introduction To Discrete Mathematics

Review

If you put $n+1$ pigeons in n pigeonholes then at least one hole would have more than one pigeon.
If $n(r-1)+1$ objects are put into n boxes, then at least one of the boxes contains r or more of the objects.
If the average of n nonnegative integers $a_1, a_2, \ldots a_n$ is greater than $r-1$, i.e.,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} > r - 1,$$

then at least one of the integers is greater than or equal to r.
The number of r-permutations of an n-set equals

$$P(n,r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

The number of permutations of an n-set is $P(n,n) = n!$.
The number of circular r-permutations of an n-set equals

$$\frac{P(n,r)}{r} = \frac{n!}{(n-r)!r}.$$

The number of circular permutations of an n-set is equal to $(n-1)!$
The number of r-combinations of an n-set equals

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}.$$

The number of r-permutations of the multiset $\{\infty \cdot x_1, \infty \cdot x_2, \ldots, \infty \cdot x_k\}$ equals k^r.
The number of permutations of the multiset $\{n_1 \cdot x_1, n_2 \cdot x_2, \ldots, n_k \cdot x_k\}$ equals

$$\frac{n!}{n_1! n_2! \cdots n_k!}, \qquad \text{where } n = n_1 + n_2 + \cdots + n_k$$

Then the number of r-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$ (the number of r-combinations with repetition allowed) equals $\binom{k+r-1}{r} = \binom{k+r-1}{k-1}$.
The number of nonnegative integer solutions for the equation $x_1 + x_2 + \cdots + x_k = r$ equals $\binom{k+r-1}{r} = \binom{k+r-1}{k-1}$.
The number of positive integer solutions for the equation $x_1 + x_2 + \cdots + x_k = r$ equals $\binom{r-1}{k-1}$.
The number of ways to place r identical balls into k distinct boxes equals $\binom{k+r-1}{r} = \binom{k+r-1}{k-1}$.
The number of ways to place r identical balls into k distinct boxes such that no box remains empty equals $\binom{r-1}{k-1}$.

Algorithm for generating the permutations of $\{1, 2, \ldots, n-1, n\}$:
Begin with $\overleftarrow{1}\,\overleftarrow{2} \cdots \overleftarrow{n}$.
While there exists a mobile integer, do
(1) Find the largest mobile integer m
(2) Switch m and the adjacent integer its arrow points to.
(30 Switch thew direction of all the arrows above integers p with $p > m$.

Algorithm 1 for construction of a permutation from its inversion sequence $(a_1, a_2, \ldots, a_n)$:
(n) Write down n.
...
(n-k) Insert $n-k$ to the right of the a_{n-k}th existing number
...

Algorithm 2 for construction of a permutation from its inversion sequence $(a_1, a_2, \ldots, a_n)$:
(0) Mark down n empty spaces.
For $k = 1$ till n
Put k into the $a_k + 1$st empty space from the left.

Algorithm for generating combinations of $\{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}$:
Begin with $a_{n-1}a_{n-2} \cdots a_1 a_0 = 00 \ldots 00$.
While $a_{n-1}a_{n-2} \cdots a_1 a_0 \neq 11 \ldots 11$, do
(1) Find the smallest integer j such that $a_j = 0$.
(2) Replace a_j by 1 and each of $a_{j-1}, \ldots, a_1, a_0$ by 0.
The algorithm stops when $a_{n-1}a_{n-2} \cdots a_1 a_0 = 11 \ldots 11$.

Algorithm for generating reflected Gray codes of order n:
Begin with $a_{n-1}a_{n-2} \cdots a_1 a_0 = 00 \ldots 00$.
While $a_{n-1}a_{n-2} \cdots a_1 a_0 \neq 10 \ldots 00$, do
(1) If $a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 =$ even, then change a_0 (from 0 to 1 or 1 to 0).
(2) If $a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 =$ odd, find the smallest j such that $a_j = 1$ and change a_{j+1} (from 0 to 1 or 1 to 0).

Algorithm for generating r-combinations of $S = \{1, 2, \ldots, n-1, n\}$:
Begin with $12 \cdots r$.
While $a_1 a_2 \cdots a_r \neq (n-r+1) \cdots (n-1)n$, do
(1) Find the largest integer k such that $a_k < n$ and $a_k + 1$ is not in the $a_1 a_2 \cdots a_r$.
(2) Replace $a_1 a_2 \cdots a_r$ with

$$a_1 a_2 \cdots a_{k-1}(a_k + 1)(a_k + 2) \cdots (a_k + r - k + 1).$$

Algorithm for a linear extension of an n-poset:
Step 1. Choose a minimal element x_1 from X (with respect to the ordering $\leq$).
Step 2. Delete x_1 from X; choose a minimal element x_2 from X.
Step 3. Delete x_2 from X and choose a minimal element x_3 from X.
...
Step n. Delete x_{n-1} from X and choose the only element x_n in X.

For a real α and an integer k,

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if} \quad k \geq 1 \\ 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \leq -1. \end{cases}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad (1 \leq k \leq n-1)$$

$$\binom{n}{k} = \binom{n}{n-k} \qquad (0 \leq k \leq n)$$

$$k\binom{n}{k} = n\binom{n-1}{k-1} \qquad (1 \leq n)$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \qquad (\text{ n} \geq 0)$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0 \qquad (\text{ n} \geq 1)$$

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots (= 2^{n-1}) \qquad (\text{ n} \geq 1)$$

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1} \qquad (\text{ n} \geq 1)$$

$$1^2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + n^2\binom{n}{n} = n(n+1)2^{n-2} \qquad (\text{ n} \geq 1)$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \qquad (\text{ n} \geq 0)$$

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}$$

$$\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \qquad (1 \leq k \leq n)$$

Binomial expansion. For integer $n \geq 1$ and variables x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Newton's binomial expansion. For a real α and variables x and y with $0 \leq |x| \leq |y|$,

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}.$$

Multinomial expansion. For integer $n \geq 1$ and variables $x_1, x_2, \ldots, x_k$,

$$(x_1 + x_2 + \cdots + x_t)^n = \sum_{n_1+n_2+\cdots+n_t=n; n_1, n_2, \ldots, n_t \geq 0} \binom{n}{n_1, n_2, \ldots, n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.$$

Sperner's theorem. Any clutter of an n-set S contains at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ subsets of S.
The power set $P(S)$ can be partitioned into m disjoint chains $C_1, C_2, \dots, C_{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$.
Construction of a symmetric chain partition for the case n given a symmetric chain partition for the case $n-1$: for each chain $A_1 \subset A_2 \subset \cdots \subset A_k$ for the case $n-1$: if $k \geq 2$, do $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\}$ and $A_1 \cup \{n\} \subset A_2 \cup \{n\} \subset \cdots \subset A_{k-1} \cup \{n\}$; if $k = 1$, do $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\}$.
Dilworth's theorem.

$$\min\{k : A_1 \cup \cdots \cup A_k \text{ is an antichain partition }\} = \max\{|C| : C \text{ is a chain }\}.$$

$$\min\{k : C_1 \cup \cdots \cup C_k \text{ is a chain partition }\} = \max\{|A| : A \text{ is an antichain }\}.$$

Let $P_1, P_2, \dots, P_n$ be properties of the objects of a finite set S. Let A_i be the set of all elements of S that have the property P_i. The number of objects of S that have none of the properties $P_1, P_2, \dots, P_n$ is given by

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

The number of objects of S that have at least one of the properties $P_1, P_2, \dots, P_n$ is given by

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

A permutation $i_1 i_2 \dots i_n$ of $\{1, 2, \dots, n\}$ is called a *derangement* if $i_k \neq k$ for any $1 \leq k \leq n$ (no number remains in its position).The number D_n of derangements of $\{1, 2, \dots, n\}$ is given by

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}).$$

The derangement sequence D_n satisfies the following recurrence relations

$$D_n = (n-1)(D_{n-1} + D_{n-2}), \quad D_1 = 0, D_2 = 1, \quad \text{and}$$

$$D_n = nD_{n-1} + (-1)^n, \quad D_1 = 0.$$

A permutation of $\{1, 2, \dots, n\}$ is called *nonconsecutive* if none of $12, 23, \dots, (n-1)n$ occurs. The number Q_n of nonconsecutive permutations of $\{1, 2, \dots, n\}$ is given by

$$Q_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!$$

For $n \geq 2$, $\quad Q_n = D_n + D_{n-1}$.
A circular permutation of $\{1, 2, \dots, n\}$ is called *nonconsecutive* if none of $12, 23, \dots, n1$ occurs. The number C_n of nonconsecutive circular permutations is given by

$$C_n = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k-1)! + (-1)^n.$$

Let $|X| = m$ and let $|Y| = n$. The number of all functions from X to Y equals n^m. The number of injective functions from X to Y equals $\binom{n}{m} m! = P(n, m)$. The number $S(m, n)$ of surjective functions from X to Y is given by

$$S(m, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Theorem 1. Let $q \neq 0$. The geometric sequence $h_n = q^n$ is a solution of the recurrence relation

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}; \quad a_k \neq 0, n \geq k \tag{1}$$

if and only if the number q is a root of the characteristic equation

$$x_k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_{k-1} x - a_0 = 0. \tag{2}$$

Theorem 2. If the characteristic equation (2) has k distinct roots $q_1, q_2, \ldots, q_k$, then the general solution of (1) is

$$h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n.$$

Theorem 3. Let $q_1, q_2, \ldots, q_s$ be distinct roots with the multiplicities $m_1, m_2, \ldots, m_s$ respectively for the characteristic equation (2). Then the sequences

$$\begin{array}{l} q_1^n, nq_1^n, n^2 q_1^n, \ldots, n^{m_1-1} q_1^n; \\ q_2^n, nq_2^n, n^2 q_2^n, \ldots, n^{m_2-1} q_2^n; \\ \ldots \\ q_s^n, nq_s^n, n^2 q_s^n, \ldots, n^{m_s-1} q_s^n; \end{array}$$

are linearly independent solutions of the recurrence relation (1). Their linear combinations form the general solution of the recurrence relation (1).
Theorem 4. Let h_n^* be any particular solution of the recurrence relation

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n; \quad a_k \neq 0, n \geq k, \tag{3}$$

and let $\overline{h_n}$ be the general solution of its corresponding the homogeneous recurrence relation. Then $h_n = \overline{h_n} + h_n^*$ is the general solution of the recurrence relation (3).
Consider a first-order linear nonhomogeneous recurrence relation

$$h_n = a h_{n-1} + b_n; \quad n \geq 1 \tag{4}$$

Theorem 5. Let $b_n = cq^n$. Then (4) has a particular solution of the following form:

- If $q \neq a$, then $h_n^* = Aq^n$.
- If $q = a$, then $h_n^* = Anq^n$.

Theorem 6. Let $b_n = \sum_{i=0}^{k} c_i n^i$.

- If $a \neq 1$, then (4) has a particular solution of the form $h_n^* = A_0 + A_1 n + A_2 n^2 + \cdots + A_k n^k$.
- If $a = 1$, then the solution of (4) is $h_n = h_0 + \sum_{i=0}^{k} b_i$,

Theorem 7. Given a nonhomogeneous linear recurrence relation of the second order

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + cq^n; \quad n \geq 2 \tag{5}$$

Let q_1 and q_2 be solutions of its characteristic equation $x_2 - a_1 x - a_2 = 0$. Then (6) has a particular solution of the following forms:

- If $q \neq q_1, q \neq q_2$, then $h_n^* = Aq^n$.
- If $q = q_1, q \neq q_2$, then $h_n^* = Anq^n$.

- If $q = q_1 = q_2$, then $h_n^* = An^2q^n$.

Theorem 8. Given a nonhomogeneous linear recurrence relation of the second order

$$h_n = a_1h_{n-1} + a_2h_{n-2} + b_n; \quad n \geq 2 \tag{6}$$

where b_n is a polynomial function of n with degree k.

- If $a_1 + a_2 \neq 1$, then (6) has a particular solution of the form:

  $$h_n^* = A_0 + A_1n + A_2n^2 + \cdots + A_kn^k,$$

  where the coefficients $A_0, A_1, \ldots, A_k$ are to be determined. If $k \leq 2$, then a particular solution has the form

  $$h_n^* = A_0 + A_1n + A_2n^2.$$

- If $a_1 + a_2 = 1$, then (6) can be reduced to a first order recurrence relation

  $$g_n = (a_1 - 1)g_{n-1} + b_n, \text{ where } g_n = h_n - h_{n-1} \text{ for } n \geq 1.$$

For the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots,$ its ordinary and exponential generating functions are given by

$$\begin{aligned} A(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \ldots \\ A^{(e)}(x) &= a_0 + b_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \cdots + a_k\frac{x^k}{k!} + \ldots \end{aligned}$$

$$\begin{aligned} A(x)B(x) &= \sum_{k=0}^{\infty}(\sum_{i=0}^{k} a_ib_{k-i})x^k \\ A^{(e)}(x)B^{(e)}(x) &= \sum_{k=0}^{\infty}(\sum_{i=0}^{k} \binom{k}{i} a_ib_{k-i})\frac{x^k}{k!} \end{aligned}$$

Some ordinary generating functions:

a_i	1	c^i	i	i^2	$\binom{n}{i}$	$\binom{n+i-1}{i}$	$\frac{1}{i}; a_0 = 0$
$A(x)$	$\frac{1}{1-x}$	$\frac{1}{1-cx}$	$\frac{x}{(1-x)^2}$	$\frac{x(1+x)}{(1-x)^3}$	$(1+x)^n$	$(1-x)^{-n}$	$\ln \frac{1}{1-x}$

Some exponential generating functions:

a_i	1	c^i	i	i^2	$i!$	$(n)_i$	$n^{(i)}$
$A^{(e)}(x)$	e^x	e^{cx}	xe^x	$x(x+1)e^x$	$(1-x)^{-1}$	$(1+x)^n$	$(1-x)^{-n}$

Given a coloring $c \in C$, the *stabilazor* of c is the set $G(c) = \{f \in G \mid f * c = c\}$.
Given a permutation $f \in G$, the *invariant set* of f is the set $C(f) = \{c \in C \mid f * c = c\}$.
Given a coloring $c \in C$, the *orbit* of c is the set $\bar{c} = \{f(c) \mid f \in G\}$.
Let C be the set of all k^n colorings of X into k colors. Then $|C(f)| = k^{\#(f)}$, where $\#(f)$ is the number of cycles in the disjoint cycle factorization of f.

Burnside's Lemma Suppose a group G of permutations of X acts on a set C of colorings of X. Then the number $N(G, C)$ of nonequivalent colorings in C is given by

$$N(G, C) = \frac{1}{|G|} \sum_{f \in G} |C(f)|.$$

Given a permutation $f \in G$, the *type* of f is an n-tuple $type(f) = (e_1, e_2, \ldots, e_n)$, where e_i is the number of i-cycles in a disjoint cycle factorization of f.

$$e_1 + e_2 + \cdots + e_n = \#(f), \quad 1e_1 + 2e_2 + \cdots + ne_n = n.$$

To each permutation $f \in G$ with type $type(f) = (e_1, e_2, \ldots, e_n)$ we associate a monomial

$$mon(f) = z_1^{e_1} z_2^{e_2} \ldots z_n^{e_n}$$

The *cycle index* of G is

$$P_G(z_1, z_2, \ldots, z_n) = \frac{1}{|G|} \sum_{f \in G} mon(f) = \frac{1}{|G|} \sum_{f \in G} z_1^{e_1} z_2^{e_2} \ldots z_n^{e_n}.$$

Theorem Suppose there are k colors. Let C be a set of all k^n colorings of X. Then the number $N(G, C)$ of nonequivalent colorings in C is given by

$$N(G, C) = P_G(k, k, \ldots, k).$$

Theorem (Polya) Let $\{u_1, u_2, \ldots, u_k\}$ be a set of k colors. Let C be a set of any colorings of X such that the group G of permutations of X acts on the set C. Then the generating function for the number of nonequivalent colorings in C according to the number of colors of each kind is given by

$$P_G(u_1 + \cdots u_k, u_1^2 + \cdots u_k^2, \ldots, u_1^n + \cdots u_k^n).$$