Review I

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If you put $n+1$ pigeons in n pigeonholes then at least one hole would have more than one pigeon.
If $n(r-1)+1$ objects are put into n boxes, then at least one of the boxes contains r or more of the objects.
If the average of n nonnegative integers $a_1, a_2, \ldots a_n$ is greater than $r-1$, i.e.,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} > r - 1,$$

then at least one of the integers is greater than or equal to r.
The number of r-permutations of an n-set equals

$$P(n,r) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

The number of permutations of an n-set is $P(n,n) = n!$.
The number of circular r-permutations of an n-set equals

$$\frac{P(n,r)}{r} = \frac{n!}{(n-r)!r}.$$

The number of circular permutations of an n-set is equal to $(n-1)!$
The number of r-combinations of an n-set equals

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}.$$

The number of r-permutations of the multiset $\{\infty \cdot x_1, \infty \cdot x_2, \ldots, \infty \cdot x_k\}$ equals k^r.
The number of permutations of the multiset $\{n_1 \cdot x_1, n_2 \cdot x_2, \ldots, n_k \cdot x_k\}$ equals

$$\frac{n!}{n_1! n_2! \cdots n_k!}, \qquad \text{where } n = n_1 + n_2 + \cdots + n_k$$

Then the number of r-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$ (the number of r-combinations with repetition allowed) equals $\binom{k+r-1}{r} = \binom{k+r-1}{k-1}$.
The number of nonnegative integer solutions for the equation $x_1 + x_2 + \cdots + x_k = r$ equals $\binom{k+r-1}{r} = \binom{k+r-1}{k-1}$.
The number of positive integer solutions for the equation $x_1 + x_2 + \cdots + x_k = r$ equals $\binom{r-1}{k-1}$.
The number of ways to place r identical balls into k distinct boxes equals $\binom{k+r-1}{r} = \binom{k+r-1}{k-1}$.
The number of ways to place r identical balls into k distinct boxes such that no box remains empty equals $\binom{r-1}{k-1}$.

Algorithm for generating the permutations of $\{1, 2, \ldots, n-1, n\}$:
Begin with $\overleftarrow{1}\,\overleftarrow{2} \cdots \overleftarrow{n}$.
While there exists a mobile integer, do
(1) Find the largest mobile integer m
(2) Switch m and the adjacent integer its arrow points to.
(30 Switch thew direction of all the arrows above integers p with $p > m$.

Algorithm 1 for construction of a permutation from its inversion sequence $(a_1, a_2, \ldots, a_n)$:
(n) Write down n.
...
(n-k) Insert $n-k$ to the right of the a_{n-k}th existing number
...

Algorithm 2 for construction of a permutation from its inversion sequence $(a_1, a_2, \ldots, a_n)$:
(0) Mark down n empty spaces.
For $k = 1$ till n
Put k into the $a_k + 1$st empty space from the left.

Algorithm for generating combinations of $\{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}$:
Begin with $a_{n-1}a_{n-2} \cdots a_1 a_0 = 00 \ldots 00$.
While $a_{n-1}a_{n-2} \cdots a_1 a_0 \neq 11 \ldots 11$, do
(1) Find the smallest integer j such that $a_j = 0$.
(2) Replace a_j by 1 and each of $a_{j-1}, \ldots, a_1, a_0$ by 0.
The algorithm stops when $a_{n-1}a_{n-2} \cdots a_1 a_0 = 11 \ldots 11$.

Algorithm for generating reflected Gray codes of order n:
Begin with $a_{n-1}a_{n-2} \cdots a_1 a_0 = 00 \ldots 00$.
While $a_{n-1}a_{n-2} \cdots a_1 a_0 \neq 10 \ldots 00$, do
(1) If $a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 =$ even, then change a_0 (from 0 to 1 or 1 to 0).
(2) If $a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 =$ odd, find the smallest j such that $a_j = 1$ and change a_{j+1} (from 0 to 1 or 1 to 0).

Algorithm for generating r-combinations of $S = \{1, 2, \ldots, n-1, n\}$:
Begin with $12 \cdots r$.
While $a_1 a_2 \cdots a_r \neq (n-r+1) \cdots (n-1)n$, do
(1) Find the largest integer k such that $a_k < n$ and $a_k + 1$ is not in the $a_1 a_2 \cdots a_r$.
(2) Replace $a_1 a_2 \cdots a_r$ with

$$a_1 a_2 \cdots a_{k-1}(a_k + 1)(a_k + 2) \cdots (a_k + r - k + 1).$$

Algorithm for a linear extension of an n-poset:
Step 1. Choose a minimal element x_1 from X (with respect to the ordering $\leq$).
Step 2. Delete x_1 from X; choose a minimal element x_2 from X.
Step 3. Delete x_2 from X and choose a minimal element x_3 from X.
...
Step n. Delete x_{n-1} from X and choose the only element x_n in X.

Practice Problems

1. There are n married couples. How many of the $2n$ people must be selected in order to guarantee that one has selected a married couple?
2. There are 50 people in the room. Some of them are acquainted with each other, some not. Prove that there are two persons in the room who have equal numbers of acquaintances.
3. We are given m arbitrary natural numbers $a_1, a_2, \ldots, a_m$. Prove that the sum of some consecutive numbers in the sequence is divisible by m.
4. Given 101 integers from $1, 2, \ldots, 200$, there are at least two integers such that one of them is divisible by other.
5. 10 points are positioned inside of the equilateral triangle of side 1. Prove that there are two of them at the distance at most $\frac{1}{3}$ from each other.
6. There are 30 classes and 1000 students in a school. Prove that at least one class has at least 34 students.
7. 51 points are placed, in a random way, into a square of side 1 unit. Can we prove that 3 of these points can be covered by a circle of radius $\frac{1}{7}$ units ?
8. Prove that of 6 people, either there are three, each pair of whom are aquainted, or there are three, each pair of whom are unaquainted. Prove that this is not true for 5 people.
9. Find the number of ways to arrange the 26 letters of the alphabet so that no two of the vowels a, e, i, o, and u occur next to each other?
10. Find the number of 7-digit numbers such that all digits are nonzero, distinct, and the digits 8 and 9 do not appear next to each other.
11. Twelve people, including two who do no wish to sit next to each other, are to be seated at a round table. How many circular seating plans can be made?
12. In how many ways can six men and six ladies be seated at a round table if the men and ladies are to sit in alternative seats?
13. How many shortest paths are there from one corner of a 9×8 grid to the opposite corner?
14. A comitee of 5 to be chosen from a club that has 10 men and 12 women. How many ways can the comittee be fomed if is to contain at least two women? How may ways, if in addition, Mrs. Brown refuse to serve together with her husband?
15. Find the number of permutations of the letters in the word "MISSISSIPPI".
16. In how many ways can 8 identical rooks be placed on an 8-by-8 chessboard so that no two rooks can attak one another?
17. In how many ways can 8 rooks of different color be placed on an 8-by-8 chessboard so that no two rooks can attak one another?
18. How many possibilities are there for one red rook, 3 blue rooks , and 4 yellow?

19. In how many ways can 8 identical non-attacking rooks be placed on a 12-by-12 chessboard?

20. Find the number of 8-permutations of the multiset $M = \{3a, 2b, 4c\}$.

21. Find the number of integer solutions for the equation

$$x_1 + x_2 + x_3 + x_4 = 10,$$

where $x_1 \geq 3, x_2 \geq 0, x_3 \geq -2, x_4 \geq 5$.

22. Find the number of nonnegative integer solutions for the equation

$$x_1 + x_2 + x_3 + x_4 < 19.$$

23. A bakery sells 8 different kinds of doughnuts. If the bakery has virtually unlimited supply of each kind, how many different options for a dozen of doughnuts are there? What if a box is to contain at least one of each kind of doughnuts?

24. In how many ways can 12 indistiguishable apples and 1 orange be distributed among three children in such a way that each child gets at least one piece of fruit?

25. In how many ways can 10 apples, 15 oranges, and 8 bananas be distributed among four children?

26. In how many ways can 10 apples, 15 oranges, and 8 bananas be distributed among four children in such a way that each child gets at least one piece of fruit of each kind?

27. In how many ways can 40 indistinguishable apples be distributed among three children in such a way that each child gets at least 5 apples?

28. In how many ways can 10 apples, 15 oranges, and 14 bananas be distributed among two children?

29. In how many ways can 10 apples, 15 oranges, and 14 bananas be distributed among two children in such a way that each child gets at least three pieces of fruit of each kind?

30. A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?

31. Twelve knights sit at the round table in King Arthur's court. Everyone has two enemies, and these are exactly his immediate neighbors at the table. In how many ways can the King Arthur choose five knights so that no two enemies are chosen?

32. Determine the number of r-combinations of the multiset $\{1 \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$.

33. Determine the total number of combinations (of any size) of the multiset $M = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$.

34. Determine the inversion sequence of the permutation 35168274.

35. Construct the permutation of $\{1, 2, \ldots, 8\}$ whose inversion sequence is $2, 5, 5, 0, 2, 1, 1, 0$.

36. Generate the 5-tuples of 0-s and 1-s by using the base 2 arithmetic generating scheme and identify them with combinations of $\{x_4, x_3, x_2, x_1, x_0\}$.

37. For $n = 8$, which combination comes after $\{x_6, x_5, x_3, x_1, x_0\}$ in the list of all combinations of $\{x_7, x_6, \ldots, x_1, x_0\}$?

38. Construct the reflected Gray code of order $n = 3$ using the algorithm.

39. For $n = 8$, which 8-tuples follow 10100110 and 00011111 in the reflected Gray code?

40. Generate all 3-combinations of $\{1, 2, 3, 4, 5\}$.

41. Generate all 3-permutations of $\{1, 2, 3, 4, 5\}$.

42. Draw the diagram representing the poset $(P(\{1, 2, 3\}), \subseteq)$.

43. Draw the diagram representing the poset $(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, |)$.

44. Find a linear extention of $(\{1, 2, 3, \ldots, n\}, |)$.

45. Find a linear extention of $(P(\{1, 2, 3\}), \subseteq)$.

Answers

1. $n+1$
9. $21!P(22,5)$
10. $P(9,7)-2P(8,6)$
11. $9\cdot 10!$
12. $5!6!$
13. $\binom{9+8}{8}$
14. $\binom{12}{2}\binom{10}{3}+\binom{12}{3}\binom{10}{2}+\binom{12}{4}\binom{10}{1}+\binom{12}{5}$

    $\binom{12}{2}\binom{10}{3}+\binom{12}{3}\binom{10}{2}+\binom{12}{4}\binom{10}{1}+\binom{12}{5}-\binom{11}{1}\binom{9}{2}-\binom{11}{2}\binom{9}{1}-\binom{11}{3}$
15. $\frac{11!}{1!4!4!2!}$
16. $8!$
17. $(8!)^2$
18. $8!\frac{8!}{1!3!4!}$
19. $(\binom{12}{8})^2 8!$
20. $\frac{8!}{2!2!4!}+\frac{8!}{3!1!4!}+\frac{8!}{3!2!3!}$
23. $\binom{8+12-1}{7};\binom{11}{7}$
24. $3\binom{12}{2}$
25. $\binom{13}{3}\binom{18}{3}\binom{11}{3}$
26. $\binom{9}{3}\binom{14}{3}\binom{7}{3}$
27. $\binom{25+2}{2}$
28. $11\cdot 16\cdot 15$
29. $5\cdot 10\cdot 9$
30. $\binom{8}{5}$
31. $\binom{6}{4}+\binom{7}{5}$
32. $\binom{r+k-2}{k-2}+\binom{r+k-3}{k-2}$
33. $(n_1+1)(n_2+1)\cdots(n_k+1)$
34. $2,4,0,4,0,0,1,0$
35. 48165723
37. $\{x_6,x_5,x_3,x_2\}$
38. $000\to 001\to 011\to 010\to 110\to 111\to 101\to 100$
39. $10100111, 00011101$
40. $123, 124, 125, 134, 135, 145, 234, 235, 245, 345.$
44. $E.g., 12345\ldots n$
45. $E.g., \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$