Review II

For a real α and an integer k,

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if} \quad k \geq 1 \\ 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \leq -1. \end{cases}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad (1 \leq k \leq n-1)$$

$$\binom{n}{k} = \binom{n}{n-k} \qquad (0 \leq k \leq n)$$

$$k\binom{n}{k} = n\binom{n-1}{k-1} \qquad (1 \leq n)$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \qquad (n \geq 0)$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0 \qquad (n \geq 1)$$

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots (= 2^{n-1}) \qquad (n \geq 1)$$

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1} \qquad (n \geq 1)$$

$$1^2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + n^2\binom{n}{n} = n(n+1)2^{n-2} \qquad (n \geq 1)$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \qquad (n \geq 0)$$

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}$$

$$\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \qquad (1 \leq k \leq n)$$

Binomial expansion. For integer $n \geq 1$ and variables x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Newton's binomial expansion. For a real α and variables x and y with $0 \leq |x| \leq |y|$,

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}.$$

Multinomial expansion. For integer $n \geq 1$ and variables $x_1, x_2, \ldots, x_k$,

$$(x_1 + x_2 + \cdots + x_t)^n = \sum_{n_1+n_2+\cdots+n_t=n n_1,n_2,\ldots,n_t\geq 0} \binom{n}{n_1, n_2, \ldots, n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.$$

Sperner's theorem. Any clutter of an n-set S contains at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ subsets of S.

The power set $P(S)$ can be partitioned into m disjoint chains $C_1, C_2, \ldots, C_m$, where $m = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Construction of a symmetric chain partition for the case n given a symmetric chain partition for the case $n-1$: for each chain $A_1 \subset A_2 \subset \cdots \subset A_k$ for the case $n-1$: if $k \geq 2$, do $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\}$ and $A_1 \cup \{n\} \subset A_2 \cup \{n\} \subset \cdots \subset A_{k-1} \cup \{n\}$; if $k = 1$, do $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\}$.

Dilworth's theorem.

$$\min\{k : A_1 \cup \cdots \cup A_k \text{ is an antichain partition }\} = \max\{|C| : C \text{ is a chain }\}.$$

$$\min\{k : C_1 \cup \cdots \cup C_k \text{ is a chain partition }\} = \max\{|A| : A \text{ is an antichain }\}.$$

Let $P_1, P_2, \ldots, P_n$ be properties of the objects of a finite set S. Let A_i be the set of all elements of S that have the property P_i. The number of objects of S that have none of the properties $P_1, P_2, \ldots, P_n$ is given by

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| = |S| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

The number of objects of S that have at least one of the properties $P_1, P_2, \ldots, P_n$ is given by

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

A permutation $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ is called a *derangement* if $i_k \neq k$ for any $1 \leq k \leq n$ (no number remains in its position).The number D_n of derangements of $\{1, 2, \ldots, n\}$ is given by

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}).$$

$$D_1 = 0, \quad D_2 = 1, \quad D_3 = 2, \quad D_4 = 9, \quad D_5 = 44, \quad D_6 = 265.$$

$$\lim_{n \to \infty} \frac{D_n}{n!} = e^{-1}.$$

The derangement sequence D_n satisfies the following recurrence relations

$$D_n = (n-1)(D_{n-1} + D_{n-2}), \quad D_1 = 0, D_2 = 1, \quad \text{and}$$

$$D_n = nD_{n-1} + (-1)^n, \quad D_1 = 0.$$

A permutation of $\{1, 2, \ldots, n\}$ is called *nonconsecutive* if none of $12, 23, \ldots, (n-1)n$ occurs. The number Q_n of nonconsecutive permutations of $\{1, 2, \ldots, n\}$ is given by

$$Q_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!$$

For $n \geq 2$, $\quad Q_n = D_n + D_{n-1}$.

A circular permutation of $\{1, 2, \ldots, n\}$ is called *nonconsecutive* if none of $12, 23, \ldots, n1$ occurs. The number C_n of nonconsecutive circular permutations is given by

$$C_n = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k-1)! + (-1)^n.$$

Let $|X| = m$ and let $|Y| = n$. The number of all functions from X to Y equals n^m. The number of injective functions from X to Y equals $\binom{n}{m}m! = P(n,m)$. The number $S(m,n)$ of surjective functions from X to Y is given by

$$S(m,n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Practice Problems

1. Find the coefficient of x^3 in the expansion of $(2+5x)^6$.
2. Find the coefficient of $x_1^3 x_2 x_3^2$ in the expansion of $(2x_1 - 3x_2 + 5x_3)^6$.
3. Find symmetric chain partition for $P(S)$, where S has one, two, three or four elements.
4. Find clutters of maximal size for $P(\{1,2,3,4,5\})$.
5. Prove that there is only one maximal clutter for $P(\{1,2,3,4\})$.
6. Consider the poset $(\{1,2,\ldots,12\},|)$:
   (a) determine a chain of the largest size and a a partition of X into the smallest number of antichains;
   (b) determine an antichain of the largest size and a a partition of X into the smallest number of chains.
7. Determine the number of 10-combinations of the multiset $M = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.
8. Find the number of integer solutions for the equation

   $$x_1 + x_2 + x_3 + x_4 = 15,$$

   where $2 \le x_1 \le 6, \quad -2 \le x_2 \le 1, \quad 0 \le x_3 \le 6, \quad 3 \le x_4 \le 8.$
9. How many ways can a hatcheck girl hand back the 10 hats of 10 gentlemen, one to each gentleman, with no man getting his hat?
10. Determine the number of permutations of $\{1,2,\ldots,n\}$ in which no odd integer is in its natural position.
11. How many ways are there to rearrange 10 camels in a caravan. so that every camel has a different camel in front of it (the position of the original first camel is arbitrary)?
12. How many ways are there for 8 children on a merry-go-round to change places so that somebody new is in front of each child? (The seats are indistinguishable and in a circle.)
13. Solve the recurrence relation $h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}$, with $h_0 = 1, h_1 = 2$, $h_2 = 0$.
14. Solve the recurrence relation $h_n = 4h_{n-1} - 4h_{n-2}$, with $h_0 = a$, $h_1 = b$.
15. Solve the recurrence relation $h_n = 3h_{n-1} - 4n$, with $h_0 = 2$.

16. Solve the recurrence relation $h_n = h_{n-1} - 3n^2 - 5n^3$, with $h_0 = 2$.

17. Solve the recurrence relation $h_n = 10h_{n-1} - 25h_{n-2} + 5^{n+1}$, with $h_0 = 5, h_1 = 15$.

18. Solve the recurrence relation $h_n = 6h_{n-1} - 9h_{n-2} + 8n^2 - 24n$, with $h_0 = 5, h_1 = 5$.

19. Find the coefficient of x^6 in $(x^2 + x^3 + x^4 + x^5 + \dots)^2$.

20. Use (ordinary) generating fuctions to find the number of ways distribute r jelly beans among 8 children if

    (a) each child gets at least one jelly bean;

    (b) each child gets at even number of beans.

21. Find the number of nonnegative integer solutions for the equation

$$y_1 + 2y_2 = n.$$

22. Find the closed form for the (ordinary) generating function of the sequence $a_i = \frac{1}{i}$.

23. Solve the recurrence relation $a_n = 2a_{n-1} + a_{n-2}, a_0 = 1, a_1 = 3$.

24. Use $\frac{(1-x^2)^n}{(1-x)^n}$ to evaluate the sum $\sum_{i=0}^{\frac{m}{2}} (-1)^i \binom{n}{i} \binom{n+m-2i-1}{n-1}$, if $m \leq n$ and m is even.

25. Find the closed form for the exponential generating function of the sequence $a_i = \frac{1}{i+1}$.

26. Use exponential generating functions to find the number of k-permutations of the multiset $\{\infty \cdot x_1, \infty \cdot x_2, \dots, \infty \cdot x_n\}$.

27. Determine the number of ways to color the squares of a 1-by-n chessboard using the colors, red, white, and blue, if an even number of squares are colored red and there is at least one blue square.