Chapter 13

Symbolic Integration

This chapter contains the basic tricks of the "symbolic integration" trade. The goal of this chapter is not to make you a slow innacurate integration software, but rather to help you understand the basics well enough to use modern integration software effectively.

The basic methods of integration are important, but there are many more tricks that are useful in special situations. The computer knows most of the special tricks but sometimes a basic preliminary hand computation allows the computer to calculate a very complicated integral that it cannot otherwise do. A change of variables often clarifies the meaning of an integral. Integration by parts is theoretically important in both math and physics.

We will encounter a few of the special tricks later when they arise in important contexts. The cable of a suspension bridge can be described by a differential equation, which can be antidifferentiated with the "hyperbolic cosine." If you decide to work on that project, you will want to learn that trick. "Partial fractions" is another integration trick that arises in the logistic growth model, the S-I-S disease model, and the linear air resistance model in a basic form. We will take that method up when we need it.

The Fundamental Theorem of Integral Calculus 12.13 gives us an indirect way to exactly compute the limit of approximations by sums of the form

$$f[a]\,\Delta x + f[a+\Delta x]\,\Delta x + f[a+2\Delta x]\,\Delta x + \cdots + f[b-\Delta x]\,\Delta x = \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [f\,[x]\,\Delta x]$$

We have

$$\int_a^b f[x]\ dx = \lim_{\Delta x\to 0} \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [f\,[x]\,\Delta x] \approx \sum_{\substack{x=a \\ \text{step } \delta x}}^{b-\delta x} [f\,[x]\,\delta x]$$

but the limit can be computed without forming the sum. The Fundamental Theorem says:

If we can find $F[x]$ so that $dF[x] = f[x]\,dx$, for all $a \le x \le b$, then

$$\int_a^b f[x]\ dx = \lim_{\Delta x\to 0} \sum_{\substack{x=a \\ \text{step } \Delta x}}^{b-\Delta x} [f\,[x]\,\Delta x] = F[b] - F[a]$$

Finding an "antiderivative" lets us skip from the left side of the above equations to the right side without going through the limit in the middle.

This indirect computation of the integral works any time we can find a trick to figure out an antiderivative. There are many such tricks, or "techniques," and the computer knows them all. Your main task in this chapter is to understand the fundamental techniques and their limitations, rather than to develop skill at very elaborate integral computations.

The rules of integration are more difficult than the rules of differentiation because they amount to trying to use the rules of differentiation in reverse. You must learn all the basic techniques to understand this, but we will not wallow very deep into the swamp of esoteric techniques. That is left for the computer.

13.1 Indefinite Integrals

The first half of the Fundamental Theorem means that we can often find an integral in two steps: 1) Find an antiderivative, 2) compute the difference in values of the antiderivative. It makes no difference which antiderivative we use. We formalize notation that breaks up these two steps.

One antiderivative of $3x^2$ is x^3, since $\frac{dx^3}{dx} = 3x^2$, but another antiderivative of $3x^2$ is $x^3 + 273$, since the derivative of the constant 273 (or any other constant) is zero. The integral $\int_a^b 3x^2 dx$ may be computed with either antiderivative.

$$\begin{aligned}\int_a^b 3x^2 dx &= x^3|_a^b = b^3 - a^3 \\ &= [x^3 + 273]|_a^b = [b^3 + 273] - [a^3 + 273] = b^3 - a^3\end{aligned}$$

Our next result is the converse of the result that says the derivative of a constant is zero. It says that if the derivative is zero, the function must be constant. This is geometrically obvious - draw the graph of a function with a zero derivative!

Theorem 13.1 *The Zero-Derivative Theorem*
Suppose $F[x]$ and $G[x]$ are both antiderivatives for the same function $f[x]$ on the interval $[a,b]$; that is, $\frac{dF}{dx}[x] = \frac{dG}{dx}[x] = f[x]$ for all x in $[a,b]$. Then $F[x]$ and $G[x]$ differ by a constant for all x in $[a,b]$.

PROOF:

The function $H[x] = F[x] - G[x]$ has zero derivative on $[a,b]$. We have

$$H[X] - H[a] = \int_a^X 0\ dx = 0$$

by the first half of the Fundamental Theorem and direct computation of the integral of zero.

If X is any real value in $[a,b]$, $H[X] - H[a] = 0$. This means $H[X] = H[a]$, a constant. In turn, this tells us that $F[X] - G[X] = F[a] - G[a]$, a constant for all X in $[a,b]$.

Definition 13.1 *Notation for the Indefinite Integral*
The indefinite integral of a function $f[x]$ *denoted*

$$\int f[x]\,dx$$

is equal to the collection of all functions $F[x]$ *with differential*

$$dF[x] = f[x]\,dx$$

or derivative $\frac{dF}{dx}[x] = f[x]$.

We write "$+c$" after an answer to indicate all possible antiderivatives. For example,

$$\int \mathrm{Cos}[\theta]\ d\theta = \mathrm{Sin}[\theta] + c$$

Exercise Set 13.1 *Verify that both* x^2 *and* $x^2 + \pi$ *equal* $\int 2x\ dx$.

13.2 Specific Integral Formulas

$$\int x^p \; dx = \frac{1}{p+1} x^{p+1} + c, \quad p \neq -1$$
$$\int \frac{1}{x} \; dx = \mathrm{Log}[x] + c, \quad x > 0$$
$$\int e^x \; dx = e^x + c$$
$$\int \mathrm{Sin}[x] \; dx = -\mathrm{Cos}[x] + c$$
$$\int \mathrm{Cos}[x] \; dx = \mathrm{Sin}[x] + c$$

Example 13.1 *Guess and Correct* x^6

Suppose we want to find

$$\int 3\, x^5 \; dx = F[x]?$$

We know that if we differentiate a power function, we reduce the exponent by 1, so we guess and check our answer,

$$F_1[x] = x^6 \qquad F_1'[x] = 6\, x^5$$

The constant is wrong but would be correct if we chose

$$F[x] = \tfrac{1}{2}\; x^6 \qquad F'[x] = \tfrac{1}{2} \cdot 6\, x^5 = 3\, x^5$$

Here is another example of guessing and correcting the guess by adjusting a constant.

Example 13.2 *Guess and Correct* $\mathrm{Sin}[3\,x]$

Find

$$\int 7\; \mathrm{Cos}[3\,x] \; dx = G[x]$$

Begin with the first guess and check

$$G_1[x] = \mathrm{Sin}[3\,x] \qquad G_1'[x] = 3\; \mathrm{Cos}[3x]$$

Adjusting our guess gives

$$G[x] = \frac{7}{3}\,\mathrm{Sin}[3\,x] \qquad G'[x] = \frac{7}{3}\,3\ \mathrm{Cos}[3x] = 7\ \mathrm{Cos}[3\,x]$$

It is best to check your work in any case so you only need to remember the five specific basic formulas above and use them to adjust your guesses. We will learn general rules based on each of the rules for differentiation but used in reverse. Here is some basic drill work.

Exercise Set 13.2

1. *Basic Drill on Guessing and Correcting*

$a)\quad \int 7\sqrt{x}\ dx = ?$	$b)\quad \int 5\,x^3\ dx = ?$
$c)\quad \int \frac{3}{x^2}\ dx = ?$	$d)\quad \int x^{\frac{3}{2}}\ dx = ?$
$e)\quad \int (5-x)^2\ dx = ?$	$f)\quad \int \mathrm{Sin}[3\,x]\ dx = ?$
$g)\quad \int e^{2\,x}\ dx = ?$	$h)\quad \int \frac{-7}{x}\ dx = ?$

You cannot do anything you like with indefinite integrals and expect to get the intended function. In particular, the "dx" in the integral tells you the variable of differentiation for the intended answer.

2. *Explain what is wrong with the following nonsense:*

$$\begin{aligned}\int x^2\ dx &= \int x\cdot x\ dx = x\int x\ dx\\ &= x\,[\frac{1}{2}\,x^2 + c] = \frac{1}{2}\,x^3 + c\,x\end{aligned}$$

and

$$\begin{aligned}\int_0^1 x^2\ dx &= \int_0^1 x\cdot x\ dx = x\int_0^1 x\ dx\\ &= x\,[\frac{1}{2}\,x^2|_0^1] = \frac{1}{2}\,x\end{aligned}$$

13.3 Superposition of Antiderivatives

$$\int a\,f[x] + b\,g[x]\ dx = a \int f[x]\ dx + b \int g[x]\ dx$$

Example 13.3 *Superposition of Derivatives in Reverse*

We prove the superposition rule

$$\int a\,f[x] + b\,g[x]\ dx = a \int f[x]\ dx + b \int g[x]\ dx$$

by letting $F[x] = \int f[x]\ dx$, $G[x] = \int g[x]\ dx$ and writing out what the claim for indefinite integrals means in terms of these functions:

$$\begin{aligned}
&F'[x] = f[x] \\
&G'[x] = g[x] \\
&(a\,F[x] + b\,G[x])' = a\,F'[x] + b\,G'[x] = a\,f[x] + b\,g[x] \\
&(a \int f[x]\ dx + b \int g[x]\ dx)' = a\,f[x] + b\,g[x]
\end{aligned}$$

and

$$(\int a\,f[x] + b\,g[x]\ dx)' = a\,f[x] + b\,g[x]$$

Do the arbitrary constants of integration matter? No, as long as we interpret the sum of two arbitrary constants as just another arbitrary constant.

Example 13.4 *Superposition for Integrals*

$$\int (2\,\mathrm{Sin}[x] - 3\frac{1}{x})\ dx = 2 \int \mathrm{Sin}[x]\ dx - 3 \int \frac{1}{x}\ dx = -2\,\mathrm{Cos}[x] - 3\,\mathrm{Log}[x] + c$$

Now, use your rule to break linear combinations of integrands into simpler pieces.

Exercise Set 13.3

1. *Superposition of Antiderivatives Drill*

   a) $\int 5\,x^3 - 2\;dx = 5\int x^3\;dx - 2\int 1\;dx =?$

   b) $\int \frac{5}{x^3} - 2\sqrt{x}\;dx = 5\int \frac{1}{x^3}\;dx - 2\int \sqrt{x}\;dx =?$

   c) $\int \frac{3}{x^2} - \sqrt[3]{x} + \frac{1}{\sqrt{x}}\;dx =?$

   d) $\int 5\,\mathrm{Sin}[x] - e^{2\,x}\;dx =?$

   e) $\int \mathrm{Sin}[5\,x] - 5\,\mathrm{Sin}[x]\;dx =?$

   f) $\int \mathrm{Cos}[5\,x] - \frac{5}{x}\;dx =?$

   Remember that the computer can be used to check your work on basic skills.

2. *Run the computer program SymbolicIntegr, and use the computer to check your work from the previous exercise.*

13.4 "Substitution" for Integrals

One way to find an indefinite integral is to change the problem into a simpler one. Of course, you want to change it into an equivalent problem.

Change of variables can be done legitimately as follows. First, let $u =$ part of the integrand. Next, calculate $du = \cdots$. If the remaining part of the integrand is du, make the substitution and; if it is not du, try a different substitution. The point is that we must look for both an expression and its differential. Here is a very simple example.

Example 13.5 *A Change of Variable and Differential*

Find

$$\int 2\,x\;\sqrt{1+x^2}\;dx = \int \sqrt{[1+x^2]}\{2\,x\,dx\}$$

Begin with

$$u = [1+x^2] \qquad du = \{2\,x\,dx\}$$

We replace the expression for u and du, thereby obtaining the simpler problem: Find

$$\begin{aligned}\int \sqrt{[u]}\ \{du\} &= \int u^{\frac{1}{2}}\ du \\ &= \frac{1}{1+\frac{1}{2}}\ u^{1+\frac{1}{2}} + c \\ &= \frac{2}{3}\ u^{\frac{3}{2}} + c\end{aligned}$$

The expression $\frac{2}{3}\ u^{\frac{3}{2}} + c$ is not an acceptable answer to the question, "What functions of x have derivative $2\,x\ \sqrt{1+x^2}$?" However, if we remember that $u = 1 + x^2$, we can express the answer as

$$\frac{2}{3}\ u^{\frac{3}{2}} + c = \frac{2}{3}\ [1+x^2]^{\frac{3}{2}} + c$$

Checking the answer will show why this method works. We use the Chain Rule:

$$y = \tfrac{2}{3}\ u^{\frac{3}{2}} \qquad\qquad u = 1 + x^2$$

so

$$\tfrac{dy}{du} = \tfrac{2}{3}\,\tfrac{3}{2}\ u^{\frac{3}{2}-1} \qquad \tfrac{du}{dx} = 2x$$

and

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = u^{\frac{1}{2}}\ 2x = 2x\sqrt{1+x^2}$$

or

$$dy = 2\,x\ \sqrt{1+x^2}\ dx$$

Example 13.6 *Another Change*

You *must* substitute for the differential du associated with your change of variables $u = \cdots$. Sometimes this is a little complicated. For example, suppose we try to compute

$$\int 2\,x\ \sqrt{1+x^2}\ dx$$

with the change of variable and differential

$$v = \sqrt{1+x^2} \qquad dv = \tfrac{x}{\sqrt{1+x^2}}\ dx$$

Our integral becomes

$$\int 2v[\text{ rest of above}]$$

with the rest of the above equal to $x\,dx$. We can find $dv = \frac{x}{\sqrt{1+x^2}}dx$ by multiplying numerator and denominator by $\sqrt{1+x^2}$, so our integral becomes

$$\begin{aligned}\int 2\sqrt{1+x^2}\,x\,\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\;dx &= \int 2\left(\sqrt{1+x^2}\right)^2 [\frac{x}{\sqrt{1+x^2}}\;dx] \\ &= 2\int v^2 dv = \frac{2}{3}\,v^3 + c \\ &= \frac{2}{3}\left[1+x^2\right]^{\frac{3}{2}}\end{aligned}$$

This is the same answer as before, but the substitution was more difficult on the dv piece.

Example 13.7 *A Failed Attempt*

Sometimes an attempt to simplify an integrand by change of variables will lead you to either a more complicated integral or a situation in which you cannot make the substitution for the differential. In these cases, scratch off your work and try another change.

Suppose we try a grand simplification of

$$\int 2\,x\;\sqrt{1+x^2}\;dx$$

taking

$$w = x\;\sqrt{1+x^2} \qquad dw = \left(\sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}}\right)\,dx$$

We might substitute w for the whole integrand, but there is nothing left to substitute for dw and we cannot complete the substitution. We simply have to try a different method.

Example 13.8 *A Less Obvious Substitution with $u = \sqrt{x}$*

We may be slipping into the symbol swamp, but a little wallowing can be fun. Here is a change of variable and differential with a twist:

$$\int \frac{\sqrt{x}}{1+x}\, dx$$

$$u = \sqrt{x} \quad \Leftrightarrow \quad u^2 = x \qquad (x > 0)$$

$$du = \frac{1}{2\sqrt{x}}\, dx \quad \Leftrightarrow \quad 2\,u\; du = dx$$

$$\int \frac{u}{1+u^2}\, 2\ u\ du = 2\int \frac{u^2}{1+u^2}\, du = 2\left[\int \left(1 - \left(\frac{1}{1+u^2}\right)\right) du\right]$$

$$= 2\int du - 2\int \frac{1}{1+u^2}\, du = 2\,u - 2\ \mathrm{ArcTan}[u] + c$$

$$\int \frac{\sqrt{x}}{1+x}\, dx = 2\sqrt{x} - 2\ \mathrm{ArcTan}[\sqrt{x}] + c$$

Now, you try it.

Exercise Set 13.4

Change of Variable and Differential Drill

a) $\int \frac{1}{(3x-2)^2}\, dx = ?$ *b*) $\int \frac{2t}{\sqrt{1-t^2}}\, dt = ?$

c) $\int x(3+7x^2)^3\, dx = ?$ *d*) $\int \frac{3y}{(2+2y^2)^2}\, dy = ?$

e) $\int (\mathrm{Cos}[x])^3\, \mathrm{Sin}[x]\, dx = ?$ *f*) $\int \frac{\mathrm{Cos}[\mathrm{Log}[x]]}{x}\, dx = ?$

g) $\int e^{\mathrm{Cos}[\theta]}\, \mathrm{Sin}[\theta]\, d\theta = ?$ *h*) $\int \mathrm{Sin}[a\,x+b]\, dx = ?$

If you have checked several indefinite integration problems that you computed with a change of variable and differential, you probably realize that the Chain Rule lies behind the method. Perhaps you can formalize your idea.

Problem 13.1 THE CHAIN RULE IN REVERSE

Use the Chain Rule for differentiation to prove the indefinite integral form of "Integration by Substitution." Once you have this indefinite rule, use the Fundamental Theorem to prove the definite rule.

13.5 Change of Limits of Integration

When we want an antiderivative such as $\int 2\,x\,\sqrt{1+x^2}\;dx$, we have no choice but to re-substitute the expressions for new variables back into our answer. In the first example of the previous section,

$$\int 2\,x\,\sqrt{1+x^2}\;dx = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}[1+x^2]^{\frac{3}{2}}$$

However, when we want to compute a definite integral such as

$$\int_5^7 2\,x\,\sqrt{1+x^2}\;dx$$

we can change the limits of integration along with the change of variables. For example,

$$\begin{aligned} u &= 1+x^2 & \qquad du &= 2\,x\,dx \\ u[5] &= 26 & \qquad u[7] &= 50 \end{aligned}$$

so the new problem is to find

$$\int_{26}^{50} u^{\frac{1}{2}}\;du$$

which equals

$$\frac{2}{3}u^{\frac{3}{2}}|_{26}^{50} = \frac{2}{3}[50^{\frac{3}{2}} - 26^{\frac{3}{2}}] \approx 235.702 - 88.383 = 147.319$$

We recommend that you do your definite integral changes of variable this way:

Procedure 13.1 *To compute*

$$\int_a^b f[x]\;dx$$

1. Set a portion of your integrand $f[x]$ equal to a new variable $u = u[x]$.
2. Calculate $du =?? \; dx$.
3. Calculate $u[a] = \alpha$ and $u[b] = \beta$.
4. Substitute both the function and the differential in $f[x]\;dx$ for $g[u]\;du$.
5. Compute

$$\int_\alpha^\beta g[u]\;du$$

Pay Me Now or Pay Me Later:

You *can* compute the antiderivative in terms of x and then use the original limits of integration, but there is a danger that you will lose track of the variable with which you started. If you are careful, both methods give the same answer; for example,

$$\int_5^7 2x\sqrt{1+x^2}\ dx = \frac{2}{3}[1+x^2]^{\frac{3}{2}}|_5^7 = \frac{2}{3}\left([1+7^2]^{\frac{3}{2}} - [1+5^2]^{\frac{3}{2}}\right) \approx 147.319$$

Exercise Set 13.5

1. Change of Variables with Limits Drill

a) $\int_0^1 \frac{x}{1+x^2}\ dx = ?$

b) $\int_0^{\pi/6} \mathrm{Sin}[3\theta]\ d\theta = ?$

c) $\int_2^3 \frac{2\,x}{(x^2-3)^2}\ dx = ?$

d) $\int_0^1 a\,x + b\ dx = ?$

e) $\int_0^1 \frac{3\,x^2-1}{1+\sqrt{x-x^3}}\ dx = 0$ because of the new limits.

f) $\int_0^1 x\,e^{-x^2}\ dx = ?$

g) $\int_0^{\pi/2} \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\ d\theta = ?$

2. Use the change of variable $u = \sqrt{x}$ and associated change of differential to convert $\int \sqrt{x}\ \mathrm{Cos}[\sqrt{x}]\ dx$ into a multiple of $\int u^2\ \mathrm{Cos}[u]\ du$.

13.5.1 Integration with Parameters

Change of variables can make integrals with parameters, which are important in many scientific and mathematical problems, into specific integrals. For example, suppose that ω and a are constant. Then

$$\int \mathrm{Sin}[\omega\,t]\ dt$$

$$u = \omega\,t$$

$$du = \omega\ dt \quad \Leftrightarrow \quad \frac{1}{\omega}\ du = dt$$

$$\int \mathrm{Sin}[\omega\,t]\ dt = \int \mathrm{Sin}[u]\ \frac{1}{\omega}\ du = \frac{1}{\omega}\int \mathrm{Sin}[u]\ du$$

and

$$\int \frac{1}{a^2+x^2}\,dt = \int \frac{1}{a^2(1+(x/a)^2)}\,dx = \frac{1}{a^2}\int \frac{1}{1+(x/a)^2}\,dx$$
$$u = \frac{x}{a}$$
$$du = \frac{1}{a}\,dx \quad \Leftrightarrow \quad a\,du = dx$$
$$\int \frac{1}{a^2+x^2}\,dt = \frac{1}{a^2}\int \frac{1}{1+u^2}\,a\,du = \frac{1}{a}\int \frac{1}{1+u^2}\,du$$

This type of change of variable and differential may be the most important kind for you to think about because the computer can give you specific integrals, but you may want to see how an integral depends on a parameter.

Problem 13.2 Pull the Parameter Out

Show that

$$\int_0^3 \sqrt{9-x^2}\,dx = 3\int_0^3 \sqrt{1-(x/3)^2}\,dx = 9\int_0^1 \sqrt{1-u^2}\,du$$

Show that the area of a circle of radius r is r^2 times the area of the unit circle. First, change variables to obtain the integrals below and then read on to interpret your computation.

$$\int_0^r \sqrt{r^2-x^2}\,dx = r^2\int_0^1 \sqrt{1-u^2}\,du$$

13.6 Trig Substitutions

Study this section if you have a personal need to compute integrals with one of the following expressions (and do not have your computer):

$$\sqrt{a^2-x^2}, \qquad a^2+x^2 \quad \textit{or} \quad \sqrt{a^2+x^2}$$

You should skim read this section even if you do not wish to develop this skill because the positive sign needed to go between

$$(\mathrm{Cos}[\theta])^2 = 1-(\mathrm{Sin}[\theta])^2 \qquad \text{and} \qquad \mathrm{Cos}[\theta] = \sqrt{1-(\mathrm{Sin}[\theta])^2}$$

can cause errors in the use of a symbolic integration package. In other words, the computer may use the symbolic square root when you intend for it to use the negative.

Example 13.9 *A Trigonometric Change of Variables*

The following integral comes from computing the area of a circle using the integral. A sine substitution makes it one we can antidifferentiate with two tricks from trig.

$$\int_0^1 \sqrt{1-u^2}\, du = \int_0^{\pi/2} \sqrt{1-\mathrm{Sin}^2[\theta]}\,\mathrm{Cos}[\theta]\ \ d\theta$$

because we take the substitution

$$u = \mathrm{Sin}[\theta] \qquad du = \mathrm{Cos}[\theta]\, d\theta$$

$$u = 0 \Leftrightarrow \theta = 0 \qquad u = 1 \Leftrightarrow \theta = \pi/2$$

We know from high school trig that $1 - \mathrm{Sin}^2[\theta] = \mathrm{Cos}^2[\theta]$, so, when the cosine is positive, $\sqrt{1-\mathrm{Sin}^2[\theta]} = \mathrm{Cos}[\theta]$ and

$$\int_0^{\pi/2} \sqrt{1-\mathrm{Sin}^2[\theta]}\,\mathrm{Cos}[\theta]\, d\theta = \int_0^{\pi/2} \mathrm{Cos}^2[\theta]\, d\theta$$

We also know from high school trig (or looking at the graph and thinking a little) that $\mathrm{Cos}^2[\theta] = \frac{1}{2}[1+\mathrm{Cos}(2\theta)]$, so

$$\begin{aligned}\int_0^{\pi/2} \mathrm{Cos}^2[\theta]\, d\theta &= \frac{1}{2}[\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \mathrm{Cos}(2\theta)\, d\theta] \\ &= \frac{\pi}{4} + \frac{1}{4}\int_0^{\pi} \mathrm{Cos}[\phi]\, d\phi\end{aligned}$$

because we use another change of variables

$$\phi = 2\theta \qquad d\phi = 2d\theta$$

$$\phi = 0 \Leftrightarrow \theta = 0 \qquad \phi = \pi \Leftrightarrow \theta = \pi/2$$

Finally,

$$\frac{1}{4}\int_0^{\pi} \mathrm{Cos}[\phi]\, d\phi = \frac{1}{4}[\mathrm{Sin}[\phi]\,|_0^{\pi}\ ] = 0$$

as we could have easily seen by sketching a graph of $\mathrm{Cos}[\phi]$ from 0 to π, with equal areas above and below the ϕ-axis.

Putting all these computations together, we have

$$\int_0^1 \sqrt{1-u^2}\, du = \frac{\pi}{4}$$

One explanation why the change of variables in the previous example works is that sine and cosine yield parametric equations for the unit circle. More technically speaking, the (Pythagorean Theorem) identity

$$(\text{Sin}[\theta])^2 + (\text{Cos}[\theta])^2 = 1 \qquad \Leftrightarrow \qquad (\text{Cos}[\theta])^2 = 1 - (\text{Sin}[\theta])^2$$

becomes the identity $(\text{Cos}[\theta])^2 = 1 - u^2$ when we let $u = \text{Sin}[\theta]$. There is an important algebraic detail when we write

$$\text{Cos}[\theta] = \sqrt{1 - (\text{Sin}[\theta])^2}$$

This is false when cosine is negative. For the definite integral above, we wanted $0 \le u \le 1$ and chose $0 \le \theta \le \pi/2$ to put $\text{Sin}[\theta]$ in this range. It is also true that $\text{Cos}[\theta] \ge 0$ for this range of θ, so that

$$\text{Cos}[\theta] = \sqrt{1-u^2}$$

If the cosine were not positive in the range of interest, the integration would not be valid. More information on this difficulty is contained in the Mathematical Background Chapter on Differentiation Drill.

An expression such as $\sqrt{a^2 - x^2}$ can first be reduced to a multiple of $\sqrt{1-u^2}$ by taking $u = x/a$ and writing $\sqrt{a^2 - x^2} = a\,\sqrt{1 - (x/a)^2}$. This means that the expression $\sqrt{a^2 - x^2}$ can be converted to $\text{Cos}[\theta]$ with the substitutions $u = (x/a)^2$ and $u = \text{Sin}[\theta]$ (provided cosine is positive on the interval).

Notice that the substitution $u = \text{Cos}[\theta]$ converts $\sqrt{1-u^2}$ into $\sqrt{1 - (\text{Cos}[\theta])^2} = \text{Sin}[\theta]$, provided sine is positive. Also note that sine is positive over a different range of angles than cosine.

Another trig identity says

$$\begin{aligned}(\text{Tan}[\theta])^2 + 1 = (\text{Sec}[\theta])^2 \qquad &\Leftrightarrow \qquad \left(\frac{\text{Sin}[\theta]}{\text{Cos}[\theta]}\right)^2 + \left(\frac{\text{Cos}[\theta]}{\text{Cos}[\theta]}\right)^2 = \left(\frac{1}{\text{Cos}[\theta]}\right)^2 \\ &\Leftrightarrow \qquad (\text{Sin}[\theta])^2 + (\text{Cos}[\theta])^2 = 1\end{aligned}$$

If we make a change of variable $u = \text{Tan}[\theta]$, then $u^2 + 1$ becomes $(\text{Sec}[\theta])^2$ and $\sqrt{1+u^2} = \text{Sec}[\theta]$ provided secant is positive.

Example 13.10 *The Sine Substitution Without Endpoints*

The “cost” of not changing limits of integration in Example 13.9 is the following: The same tricks as above for indefinite integrals yield

$$\int \sqrt{1-u^2}\, du = \frac{1}{2}\,\theta + \frac{1}{4}\,\text{Sin}[\phi]$$

where $u = \text{Sin}[\theta]$ and $\phi = 2\,\theta$. If we really want the antiderivative of $\sqrt{1-u^2}$, then we must express all this in terms of u.

The first term is easy, we just use the inverse trig function

$$u = \text{Sin}[\theta] \qquad \Leftrightarrow \qquad \theta = \text{ArcSin}[u]$$

The term $\text{Sin}[\phi] = \text{Sin}[2\,\theta]$ must first be written in terms of functions of θ, (recall the addition formula for sine),

$$\text{Sin}[2\,\theta] = 2\ \text{Sin}[\theta]\ \text{Cos}[\theta]$$

Now, we use a triangle that contains the $u = \text{Sin}[\theta]$ idea. From SOH-CAH-TOA, if we take a right triangle with hypotenuse 1 and opposite side u, then $u = \text{Sin}[\theta]$ (see Figure 13.6:1). We also know that $\text{Cos}[\theta]$ is the adjacent side of this triangle. Using the Pythagorean Theorem, we have $\text{Cos}[\theta] = \sqrt{1-u^2}$

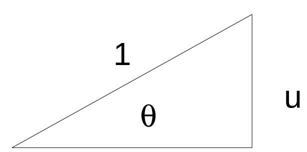

Figure 13.6:1: $\text{Sin}[\theta] = u$ and $\text{Cos}[\theta] = \text{adj/hyp} = \sqrt{1-u^2}$

Combining these facts we have,

$$\text{Sin}[2\,\theta] = 2\ \text{Sin}\,\theta]\ \text{Cos}[\theta] = 2\,u\,\sqrt{1-u^2}$$

and

$$\int \sqrt{1-u^2}\ du = \frac{1}{2}\,\theta + \frac{1}{4}\ \text{Sin}[\phi]$$
$$= \frac{1}{2}\ \text{ArcSin}[u] + \frac{u\,\sqrt{1-u^2}}{2}$$

Exercise Set 13.6

1. Geometric Proof that $\int_0^1 \sqrt{1-u^2}\ du = \frac{\pi}{4}$
   Sketch the graph $y = \sqrt{1-x^2}$ *for* $0 \leq x \leq 1$. *What geometrical shape is shown in your graph? What is the area of one fourth of a circle of unit radius?*

2. *Compute the integral* $\int_0^9 \sqrt{9-x^2}\,dx$. *(Check your symbolic computation geometrically).*

   Use the change of variable $x = \mathrm{Sin}[\theta]$ *with an appropriate differential to show that*

   $$\int_0^v \frac{1}{\sqrt{1-x^2}}\,dx = \int d\theta = \theta ... = \mathrm{ArcSin}[v]$$

   How large can we take v*?*

   Use the change of variable $x = \mathrm{Tan}[\theta]$ *with an appropriate differential to show that*

   $$\int_0^v \frac{1}{1+x^2}\,dx = \int d\theta = \theta ... = \mathrm{ArcTan}[v]$$

   How large can we take v*?*

3. Working Back from Trig Substitutions
   Suppose we make the change of variable $u = \mathrm{Sin}[\theta]$ *(as in the integration above). Express* $\mathrm{Tan}[\theta]$ *in terms of* u *by using Figure 13.6:1 and TOA.*

   A triangle is shown in Figure 13.6:2 for a change of variable $v = \mathrm{Cos}[\theta]$*. Express* $\mathrm{Sin}[\theta]$ *and* $\mathrm{Tan}[\theta]$ *in terms of* v *by using the figure and the Pythagorean Theorem.*

   A triangle is shown in Figure 13.6:2 for a change of variable $w = \mathrm{Tan}[\theta]$*. Express* $\mathrm{Sin}[\theta]$ *and* $\mathrm{Cos}[\theta]$ *in terms of* w *by using the figure and the Pythagorean Theorem.*

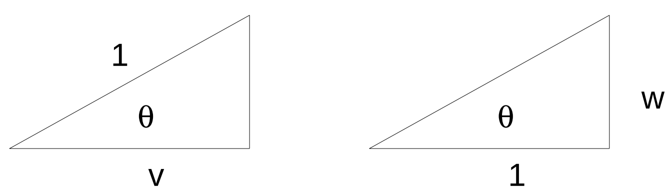

Figure 13.6:2: Triangles for $v = \mathrm{Cos}[\theta] = \mathrm{CAH}$ and $w = \mathrm{Tan}[\theta] = \mathrm{TOA}$

We are beginning to wallow a little too deeply in trig. The point of the previous example could simply be: change the limits of integration when you change variable and differential. However, it is possible to change back to u, and the previous exercise gives you a start on the trig skills needed to do this.

Problem 13.3 *A Constant Use the change of variable* $x = \mathrm{Cos}[\theta]$ *with an appropriate differential to show that*

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\int d\theta = -\theta ... = -\mathrm{ArcCos}[x] + c$$

Also, use the change of variable $x = \text{Sin}[\theta]$ *with an appropriate differential to show that*

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \int d\theta = \theta ... = \text{ArcSin}[x] + c$$

Is $\text{ArcCos}[x] = -\text{ArcSin}[x]$*? Ask the computer to Plot* $\text{ArcSin}[x]$ *and* $-\text{ArcCos}[x]$*. Why do they look alike? How do they differ? Do the graphs of* $-\text{Cos}[\theta]$ *and* $\text{Sin}[\theta]$ *look alike? How do they differ?*

13.7 Integration by Parts

Integration by Parts is important theoretically, but it appears to be just another trick. It is more than that, but first you should learn the trick.

The formulas for the "technique" are

$$\boxed{\int_a^b u\,[x]\ \ dv\,[x] = u\,[x]\,v\,[x]\,|_a^b - \int_a^b v\,[x]\ \ du\,[x]}$$

or, suppressing the x dependence,

$$\boxed{\int_{x=a}^b u\ dv = uv|_{x=a}^b - \int_{x=a}^b v\ du}$$

The idea is to break up an integrand into a function $u[x]$ and a differential $dv[x]$ where you can find the differential $du[x]$ (usually easy), the antiderivative $v[x]$ (sometimes harder), and, finally, where $\int v[x]\ du[x]$ is an easier problem. Unfortunately, often the only way to find out if the new problem is easier is to go through all the substitution steps for the terms.

Helpful Notation

We encourage you to block off the four terms in this formula.

First break up your integrand into u and dv

$$u = \qquad dv =$$

and then compute the differential of u and the antiderivative of dv

$$du = \qquad v =$$

Here is an example of the use of Integration by Parts.

Example 13.11 *Integration by Parts for* $\int_a^b x \operatorname{Log}[x]\, dx = ?$

Use the "parts"

$$u = \operatorname{Log}[x] \qquad\qquad dv = x\ dx$$

and

$$du = \frac{1}{x}\ dx \qquad\qquad v = \frac{1}{2}x^2$$

making the integrals

$$\int_a^b udv = uv|_a^b - \int_a^b vdu$$

$$\begin{aligned}
\int_a^b x \operatorname{Log}[x]\, dx &= \frac{1}{2}x^2\ \operatorname{Log}[x]\,|_a^b - \int_a^b \frac{1}{2}x^2 \frac{dx}{x} \\
\int_a^b x \operatorname{Log}[x]\, dx &= \frac{1}{2}x^2\ \operatorname{Log}[x]\,|_a^b - \int_a^b \frac{1}{2}x dx \\
\int_a^b x \operatorname{Log}[x]\, dx &= \frac{1}{2}x^2\ \operatorname{Log}[x]\,|_a^b - \frac{x^2}{4}|_a^b \\
\int_a^b x \operatorname{Log}[x]\, dx &= \frac{1}{2}[b^2 \operatorname{Log}(b) - a^2 \operatorname{Log}[a]] - \frac{1}{4}[b^2 - a^2]
\end{aligned}$$

provided that both a and b are positive. (Otherwise, Log is undefined).

CHECK:

Notice that the indefinite integral of the calculation above is

$$\int x \operatorname{Log}[x]\, dx = \frac{1}{2}\, x^2\ \operatorname{Log}[x] - \frac{1}{4}\, x^2 + c$$

We check the correctness of this antiderivative by differentiating the right side of the equation. First, we use the Product Rule

$$3\frac{d(f[x]\cdot g[x])}{dx} = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx}$$

$$f[x] = x^2 \qquad g[x] = \mathrm{Log}[x]$$
$$\frac{df}{dx} = 2\,x \qquad \frac{dg}{dx} = \frac{1}{x}$$

$$\begin{aligned}\frac{d(x^2\,\mathrm{Log}[x])}{dx} &= 2\,x\cdot\mathrm{Log}[x] + x^2\cdot\frac{1}{x}\\ &= 2\,x\cdot\mathrm{Log}[x] + x\end{aligned}$$

Next, we differentiate the whole expression

$$\frac{d(\frac{1}{2}\,x^2\ \mathrm{Log}[x] - \frac{1}{4}\,x^2 + c)}{dx} = x\,\mathrm{Log}[x] + \frac{1}{2}\,x - \frac{2}{4}\,x + 0 = x\ \mathrm{Log}[x]$$

This verifies that the indefinite integral is correct.

Example 13.12 $\int x\,e^x\ dx$

To compute this integral, use integration by parts with

$$u = x \qquad dv = e^x\ dx$$

so

$$du = dx \qquad v = e^x$$

This gives $\boxed{\int u\ dv = u\ v - \int v\ du\ =}$

$$\begin{aligned}\int x\,e^x\ dx &= x\,e^x - \int e^x\ dx\\ &= x\,e^x - e^x = (x-1)\,e^x + c\\ \int x\,e^x\ dx &= (x-1)\,e^x + c\end{aligned}$$

CHECK:

Differentiate using the Product Rule:

$$\frac{d(f[x]\cdot g[x])}{dx} = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx}$$

$$f[x] = (x-1) \qquad g[x] = e^x$$
$$\frac{df}{dx} = 1 \qquad \frac{dg}{dx} = e^x$$

$$\begin{aligned}\frac{d((x-1)\,e^x)}{dx} &= e^x + (x-1)\,e^x\\ &= x\,e^x\end{aligned}$$

Example 13.13 *A Reduction of One Integral to a Previous One*

The integral $\int x^2\, e^x\ dx$ can be done "by parts" in two steps. First, take the parts

$$\begin{array}{lcc} & u = x^2 & dv = e^x\ dx \\ \text{so} & & \\ & du = 2\,x\ dx & v = e^x \end{array}$$

This gives $\overline{\underline{\int u\ dv = u\ v - \int v\ du\ =}}$

$$\begin{aligned}\int x^2\, e^x\ dx &= x^2\, e^x - 2\int x\, e^x\ dx \\ &= x^2\, e^x - 2(x-1)\, e^x + c\end{aligned}$$

Note that the second integral was computed in the previous example, so

$$\int x^2\, e^x\ dx = (x^2 - 2\,x + 2)\, e^x + c$$

CHECK:

Differentiate using the Product Rule:

$$\frac{d(f[x]\cdot g[x])}{dx} = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx}$$

$$\begin{array}{ll} f[x] = (x^2 - 2\,x + 2) & g[x] = e^x \\ \dfrac{df}{dx} = 2\,x - 2 & \dfrac{dg}{dx} = e^x \end{array}$$

$$\begin{aligned}\frac{d((x^2 - 2\,x + 2)\, e^x)}{dx} &= (2\,x - 2)\, e^x + (x^2 - 2\,x + 2)\, e^x \\ &= x^2\, e^x\end{aligned}$$

Example 13.14 *Circular Parts Still Gives an Answer*

We compute the integral $\int e^{2\,x}\ \mathrm{Sin}[3\,x]\ dx$ by the parts

$$u = e^{2\,x} \qquad\qquad dv = \mathrm{Sin}[3\,x]\ dx$$

so

$$du = 2\,e^{2\,x}\ dx \qquad\qquad v = -\frac{1}{3}\,\mathrm{Cos}[3\,x]$$

This gives $\overline{\underline{\int u\ dv\ =\ u\ v\ -\ \int v\ du\ =}}$

$$\int e^{2\,x}\ \mathrm{Sin}[3\,x]\ dx = -\frac{1}{3}\,e^{2\,x}\,\mathrm{Cos}[3\,x] + \frac{2}{3}\int e^{2\,x}\,\mathrm{Cos}[3\,x]\ dx$$

Now, use the parts on the second integral,

$$w = e^{2\,x} \qquad\qquad dz = \mathrm{Cos}[3\,x]\ dx$$

so

$$dw = 2\,e^{2\,x}\ dx \qquad\qquad z = \frac{1}{3}\,\mathrm{Sin}[3\,x]$$

This gives $\overline{\underline{\int w\ dz\ =\ w\ z\ -\ \int z\ dw\ =}}$

$$\int e^{2\,x}\ \mathrm{Cos}[3\,x]\ dx = \frac{1}{3}\,e^{2\,x}\,\mathrm{Sin}[3\,x] - \frac{2}{3}\int e^{2\,x}\,\mathrm{Sin}[3\,x]\ dx$$

Substituting this into the second integral above, we obtain

$$\begin{aligned}\int e^{2\,x}\ \mathrm{Sin}[3\,x]\ dx &= -\frac{1}{3}\,e^{2\,x}\,\mathrm{Cos}[3\,x] + \frac{2}{3}\left[\frac{1}{3}\,e^{2\,x}\,\mathrm{Sin}[3\,x] - \frac{2}{3}\int e^{2\,x}\,\mathrm{Sin}[3\,x]\ dx\right]\\ &= -\frac{1}{3}\,e^{2\,x}\,\mathrm{Cos}[3\,x] + \frac{2}{9}\,e^{2\,x}\,\mathrm{Sin}[3\,x] - \frac{4}{9}\int e^{2\,x}\,\mathrm{Sin}[3\,x]\ dx\end{aligned}$$

Bringing the like integral to the left side, we obtain

$$\begin{aligned}&\int e^{2\,x}\ \mathrm{Sin}[3\,x]\ dx + \frac{9}{4}\int e^{2\,x}\ \mathrm{Sin}[3\,x]\ dx = \frac{2}{9}\,e^{2\,x}\,\mathrm{Sin}[3\,x] - \frac{1}{3}\,e^{2\,x}\,\mathrm{Cos}[3\,x] + c\\ &\frac{4+9}{4}\int e^{2\,x}\ \mathrm{Sin}[3\,x]\ dx = \frac{2}{9}\,e^{2\,x}\,\mathrm{Sin}[3\,x] - \frac{1}{3}\,e^{2\,x}\,\mathrm{Cos}[3\,x] + c\\ &\int e^{2\,x}\ \mathrm{Sin}[3\,x]\ dx = \frac{2}{13}\,e^{2\,x}\,\mathrm{Sin}[3\,x] - \frac{3}{13}\,e^{2\,x}\,\mathrm{Cos}[3\,x] + c\end{aligned}$$

Example 13.15 *A Two-Step Computation of* $\int (Cos[x])^2 \, dx$

This integral can also be computed without the trig identities in Section 13.6. Use the parts

$$u = \mathrm{Cos}[x] \qquad dv = \mathrm{Cos}[x] \; dx$$

so

$$du = -\mathrm{Sin}[x] \; dx \qquad v = \mathrm{Sin}[x]$$

which yields the integration formula

$$\begin{aligned}\int (Cos[x])^2 \; dx &= \mathrm{Sin}[x]\,\mathrm{Cos}[x] + \int (\mathrm{Sin}[x])^2 \; dx \\ &= \mathrm{Sin}[x]\,\mathrm{Cos}[x] + \int [1 - (\mathrm{Cos}[x])^2] \; dx \\ &= \mathrm{Sin}[x]\,\mathrm{Cos}[x] + \int 1 \; dx - \int (\mathrm{Cos}[x])^2 \; dx\end{aligned}$$

so $2 \int (Cos[x])^2 \; dx = x - \mathrm{Sin}[x]\,\mathrm{Cos}[x] + c$ and

$$\int (Cos[x])^2 \; dx = \frac{x}{2} - \frac{1}{2}\,\mathrm{Sin}[x]\,\mathrm{Cos}[x] + c$$

Here are some practice problems. (Remember that you can check your work with the computer).

Exercise Set 13.7

1. *Drill on Integration by Parts*

a)	$\int \theta \; \mathrm{Cos}[\theta] \; d\theta = ?$	*b*)	$\int \theta^2 \; \mathrm{Sin}[\theta] \; d\theta = ?$
c)	$\int_0^{\pi/2} \theta \; \mathrm{Sin}[\theta] \; d\theta = ?$	*d*)	$\int_0^{\pi/2} \theta \; \mathrm{Cos}[\theta] \; d\theta = ?$
e)	$\int x e^{2\,x} \; dx = ?$	*f*)	$\int x^2 e^{2\,x} \; dx = ?$
g)	$\int e^{5\,x} \; \mathrm{Sin}[3\,x] \; dx = ?$	*h*)	$\int e^{3\,x} \; \mathrm{Cos}[5\,x] \; dx = ?$

2. *Compute* $\int \mathrm{Log}[x] \; dx$ *using integration by parts with* $u = \mathrm{Log}[x]$ *and* $dv = dx$*. Check your answer by differentiation.*

3. *Check the previous indefinite integral from Example 13.14 by using the Product Rule to differentiate* $\frac{2}{13}\, e^{2\,x}\,\mathrm{Sin}[3\,x] - \frac{3}{13}\, e^{2\,x}\,\mathrm{Cos}[3\,x]$.

   Compute the integral $\int e^{2\,x}\ \mathrm{Sin}[3\,x]\ dx$ *by the parts*

   $$u = \mathrm{Sin}[3\,x] \qquad\qquad dv = e^{2\,x}\ dx$$

   so

   $$du = 3\ \mathrm{Cos}[3\,x]\ dx \qquad\qquad v = \frac{1}{2}\,e^{2x}$$

4. *Calculate* $\int (\mathrm{Sin}[x])^2\ dx$.

   There is something indefinite about these integrals.

5. *What is wrong with the equation*

   $$\int \frac{dx}{x} = \int \frac{x}{x^2} dx = -1 + \int \frac{dx}{x}$$

   when you use integration by parts with $u = x$, $dv = \frac{dx}{x^2}$, $du = dx$, *and* $v = \frac{-1}{x}$? *Subtracting* $\int \frac{dx}{x}$ *from both sides of the equations above yields*

   $$0 = -1$$

   The proof of the Integration by Parts formula is actually easy.

6. The Product Rule in Reverse
   Use the Product Rule for differentiation to prove the indefinite integral form of Integration by Parts. Notice that if $H[x]$ *is any function,* $\int dH[x] = H[x] + c$, *by definition. Let* $H = u\,v$ *and show that* $dH = u\,dv + v\,du$. *Indefinitely integrate both sides of the* dH *equation,*

   $$u \cdot v = H[x] = \int dH = \int u\,dv + \int v\,du$$

   Once you have this indefinite rule, use the Fundamental Theorem to prove the definite rule.

   Here are some tougher problems in which you need to use more than one method at a time.

7. (a) $\int x\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]\ dx = ?$
   Notice that $\int (\mathrm{Cos}[\theta])^2\ d\theta$ *is done above two ways.*

   (b) $\int \theta\,(\mathrm{Cos}[\theta])^2\ d\theta = ?$
   Use integrals from the previous drill problems.

   (c) $\int \frac{x^3}{\sqrt{x^2-1}}\ dx = ?$
   HINT: Use parts $u = x^2$ *and* $dv = \frac{x\ dx}{\sqrt{x^2-1}}$ *and compute the* dv *integral.*

(d) $\int \frac{1}{x^3}\sqrt{\frac{1}{x}-1}\; dx = ?$
Use parts $u = \frac{1}{x}$ *and* $dv = \frac{1}{x^2}\sqrt{\frac{1}{x}-1}$. *Compute the dv integral.*

(e) $\int_0^1 \mathrm{ArcTan}[x]\; dx = ?$
Use parts $u = \mathrm{ArcTan}[x]$ *and* $dv = dx$.

(f) $\int_0^1 x\,\mathrm{ArcTan}[x]\; dx = ?$
Use parts and the previous exercise.

(g) $\int_0^1 \mathrm{ArcTan}[\sqrt{x}]\; dx = ?$
Use parts $u = \mathrm{ArcTan}[\sqrt{x}]$ *and change variables in the resulting integral.*

(h) $\int_4^9 \mathrm{Sin}[\sqrt{x}]\; dx = ?$
Use parts $u = \mathrm{Sin}[\sqrt{x}]$ *and* $dv = dx$. *Then change variables with* $w = \sqrt{x}$.

(i) $\int (\mathrm{Log}[x])^2\; dx = ?$
Use the parts $u = \mathrm{Log}[x]$ *and* $dv = \mathrm{Log}[x]\; dx$. *Calculate the dv integral.*

13.8 Impossible Integrals

There are important limitations to symbolic integration that go beyond the practical difficulties of learning all the tricks. This section explains why.

Integration by parts and the change of variable and differential are important ideas for the theoretical transformation of integrals. In this chapter, we tried to include just enough drill work for you to learn the basic methods. Before the practical implementation of general antidifferentiation algorithms on computers, development of human integration skills was an important part of the training of scientists and engineers. Now, the computer can makes this skill easier to master. The skill has always had limitations.

Early in the days of calculus, it was quite impressive that integration could be used to learn many many new formulas such as the classical formulas for the area of a circle or volume of a sphere. We saw how easy it was to generalize the integration approach to the volume of a cone. However, some simple-looking integrals have no antiderivative what so ever. This is not the result of peculiar mathematical examples.

The arclength of an ellipse just means the length measured as you travel along an ellipse. Problem 14.2 asks you to find integral formulas for this arclength. Early developers of calculus must have tried very hard to compute those integrals with symbolic antiderivatives, but after more than a century of trying, Liouville proved that there is no analytical expression for that antiderivative in terms of the classical functions.

The fact that the antiderivative has no expression in terms of old functions does not mean that the integral does not exist. If you find the following integral with the computer, you will see a peculiar result:

$$\int \mathrm{Cos}[x^2]\ dx$$

The innocent-looking integral $\int \mathrm{Cos}[x^2]\ dx$ is not innocent at all. The function $\mathrm{Cos}[x^2]$ is perfectly smooth and well behaved, but it does not have an antiderivative that can be expressed in terms of known functions. The bottom line is this: Integrals are used to *define* and numerically compute important new functions in science and mathematics, even when they do not have expressions in terms of elementary functions. Functions given by integral formulas can still be differentiated just as you did in Exercise 12.8.1.

Exercise Set 13.8

Run the **SymbolicIntegr** program.
