

Chapter 5

Symbolic Increments

The main idea of differential calculus is that small changes in smooth functions are approximately linear. In Chapter 3, we saw that “most” microscopic views of graphs appear to be linear, but we want a symbolic way to predict the view in a powerful microscope without actually having to use graphical magnification. The computations in this chapter give us that prediction.

Direct computation of the microscopic “gap” is hard work, but the formulas show when the gap gets small. This guarantees that a magnified graph will appear linear before we draw the graph. The direct computations of this chapter are proofs of specific differentiation rules. The next chapter develops the rules systematically as a procedure or “calculus” of derivatives. You could just accept the specific results of this chapter and go on to Chapter 6 to learn the “rules,” but differentiation rules are actually theorems that say a local linear approximation is guaranteed by certain systematic computations. You should understand that this means the “gap” tends to zero at all “good” points.

If the gap does not go to zero at a point, we will not see a straight line when we focus our microscope there. We know from Exercise 3.2.4 that the perfectly reasonable function

$$f[x] = \sqrt{x^2 + 2x + 1}$$

does not have the local linearity property at $x = -1$. (Its graph has a kink no matter how much you magnify it.) Some less reasonable ones like Weierstrass’ function

$$W[x] = \cos[x] + \frac{\cos[3x]}{2} + \frac{\cos[3^2x]}{2^2} + \dots + \frac{\cos[3^n x]}{2^n} + \dots$$

are continuous but not locally linear at any point (see Figure 3.2.8). Calculus gives a procedure to find out if a function given by a formula is locally linear.

Once we have the basic rules of this chapter and some functional rules from the next, we will differentiate the kink function

$$f[x] = \sqrt{x^2 + 2x + 1} \quad \Rightarrow \quad f'[x] = \frac{x + 1}{\sqrt{x^2 + 2x + 1}}$$

but then notice that $f'[-1]$ is not defined, so that we cannot make any conclusion about local linearity at this point. (General differentiation rules cannot say there is a kink, only that no general conclusion is possible when the rules do not apply.) At every other x , the rules do apply and this function is locally linear.

5.1 The Gap for Power Functions

The gap ε for $y = f[x] = x^p$ with $f'[x] = px^{p-1}$

In Chapter 3, we formulated the deviation of $y = f[x]$ from a straight line geometrically. The observed error or gap between the curve and line at magnification $1/\Delta x$, denoted ε , satisfies

$$f[x + \Delta x] - f[x] = m \cdot \Delta x + \varepsilon \cdot \Delta x$$

The term $\varepsilon \cdot \Delta x$ is the actual unmagnified error that appears in the formula above. We observe ε because of magnification. The number m is the slope of the microscopic straight line $dy = m dx$ in (dx, dy) -coordinates focused over x . (See Chapter 1 for equations of lines in local coordinates.) “Approximate linearity” means that the error ε is too small to measure, $\varepsilon \approx 0$, when $\delta x \approx 0$ is “small enough.”

Notice that m may depend on x , but not Δx , because the slope of the curve depends on the point but not the magnification. As we move the focus point of the microscope, the slope may change, but the graph should always appear straight under the microscope. Since m depends on x and not on Δx , it is customary to denote this slope by $f'[x]$ rather than m . The function $f'[x]$ is called the derivative of the function $f[x]$ and the local linear equation $dy = f'[x] dx$ is called the differential of $y = f[x]$. (It is the equation of the tangent line at a fixed value of x in local (dx, dy) -coordinates.)

We will compute the symbolic gap for all the examples in this section (and Exercise 5.1.1) by the steps

Procedure 5.1 Computing the ε -Gap

- 1) Compute $\frac{f[x + \Delta x] - f[x]}{\Delta x}$
- 2) Simplify the expression from the first part and compute

$$f'[x] = \lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x}$$

Give an intuitive justification of why your limit is correct.

- 3) Use your limit $f'[x]$ to solve for $\varepsilon = \frac{f[x + \Delta x] - f[x]}{\Delta x} - f'[x]$
- 4) Show that $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$, or $\varepsilon \approx 0$ is small when $\Delta x = \delta x \approx 0$ is small.

The error ε is the “gap” or amount of deviation from straightness we see above $x + \Delta x$ at power $1/\Delta x$. We want to let Δx get “small enough” so that ε is below the resolution of our microscope. This chapter shows symbolically that ε tends to zero for various functions at “good” points.

Example 5.1 Increments of $y = f[x] = x^3$

Let $y = f[x] = x^3$ and calculate the increment corresponding to a change in x of Δx . First, we know from Example 1.4.1 that

$$\frac{f[x + \Delta x] - f[x]}{\Delta x} = 3x^2 + (3x + \Delta x)\Delta x$$

Second, the intuitive limit

$$\lim_{\Delta x \rightarrow 0} \frac{f[x+\Delta x]-f[x]}{\Delta x} = 3x^2$$

since $(\Delta x(3x + \Delta x)) = 0$ when we plug $\Delta x = 0$ into the expression above. (This is technically correct, too, as long as x is bounded by some fixed amount.)

Third,

$$\varepsilon = (3x + \Delta x)\Delta x$$

comparing the lines above.

Fourth, ε tends to zero as Δx tends to zero. Here is a rough argument that shows that the ε -gap error becomes small as Δx becomes small: If x is no more than a thousand, $|x| \leq 1000$, and we want the observed error to be less than one one millionth, $|\varepsilon| < 10^{-6}$, then Δx needs to be small enough so that $|\Delta x|$ times 3001 is less than a millionth, for example, $|\Delta x| < 10^{-10}$, because

$$|\Delta x(3x + \Delta x)| < 10^{-10}(3001) < 10^{-6} \frac{3001}{10,000} < \frac{1}{1,000,000}$$

As long as x is bounded, we can make the error as small as we please by choosing a small enough Δx . The exact formula for how small Δx needs to be is not so obvious, but it is clear that for any fixed bound on x , the error $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$ for all $|x| \leq b$.

Finally, we rewrite this in the form of the increment (or microscope) approximation:

Increment of $f[x] = x^3$:

$$\begin{aligned} (x + \Delta x)^3 - x^3 &= 3x^2\Delta x + (\Delta x(3x + \Delta x))\Delta x \\ f[x + \Delta x] - f[x] &= f'[x] \cdot \Delta x + \varepsilon \cdot \Delta x \end{aligned}$$

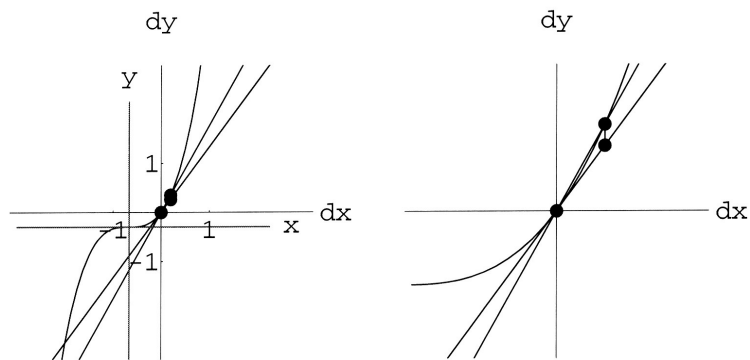
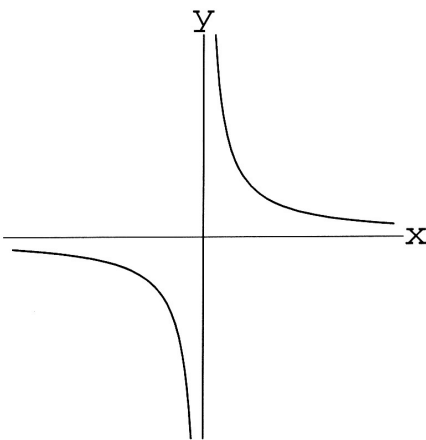
with $f'[x] = 3x^2$ and $\varepsilon = (\Delta x(3x + \Delta x))$.

We summarize the knowledge that ε can be made small by making Δx small by writing

$$y = x^3 \quad \Rightarrow \quad dy = 3x^2 dx$$

This formula, called a “differential,” omits the error term and is just the local (dx, dy) -equation at a fixed value of (x, y) for the tangent line to $y = x^3$. The complete calculation means that the nonlinear graph approaches that line as we magnify more and more; in other words, the “gap” in the microscope, ε , tends to zero for all x beneath some bound, $|x| \leq b$ (see Figure 5.1).

In the next example we calculate the gap for $y = \frac{1}{x}$ and show that there is a new complication in making ε tend to zero. There will sometimes have to be exceptional values of x . In these cases, we cannot even get close to the bad values of x . This is an annoying but necessary complication as you might suspect by looking at Figure 5.2 near zero.

Figure 5.1: The gap near $x = 2/3$ on $y = x^3$ Figure 5.2: $y = 1/x$

Example 5.2 *Exceptional Numbers and the Derivative of $y = \frac{1}{x}$*

We follow the steps of Procedure 5.1 and Exercise 5.1.

1) Compute $\frac{f[x + \Delta x] - f[x]}{\Delta x}$:

$$\begin{aligned} f[x + \Delta x] - f[x] &= \frac{1}{x + \Delta x} - \frac{1}{x} = \frac{x - (x + \Delta x)}{x(x + \Delta x)} \\ &= \frac{-1}{x(x + \Delta x)} \cdot \Delta x \\ \frac{f[x + \Delta x] - f[x]}{\Delta x} &= \frac{-1}{x(x + \Delta x)} \end{aligned}$$

2) Compute

$$f'[x] = \lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x}$$

intuitively.

It is “clear” that as $\Delta x \rightarrow 0$,

$$\lim_{\Delta x \rightarrow 0} \frac{-1}{x(x + \Delta x)} = \frac{-1}{x \cdot (x + 0)} = \frac{-1}{x^2}$$

(This is true unless $x \rightarrow 0$ at the same time. In particular, if $x = -\Delta x$, then $\frac{-1}{x(x+\Delta x)}$ is not even defined.)
At least with x fixed, we should have

$$f'[x] = -\frac{1}{x^2}$$

3) We make the “gap” error explicit by the formula

$$\varepsilon = \frac{f[x + \Delta x] - f[x]}{\Delta x} - f'[x]$$

and put the expression on a common denominator

$$\begin{aligned} \varepsilon &= \frac{-1}{x(x + \Delta x)} - \frac{-1}{x^2} \\ &= \frac{1}{x^2} + \frac{-1}{x(x + \Delta x)} \\ &= \frac{x + \Delta x}{x^2(x + \Delta x)} + \frac{-x}{x^2(x + \Delta x)} \\ &= \frac{\Delta x}{x^2(x + \Delta x)} \\ &= \Delta x \cdot \frac{1}{x^2(x + \Delta x)} \end{aligned}$$

4) Show that $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$. It seems “clear” that

$$\lim_{\Delta x \rightarrow 0} \varepsilon = \lim_{\Delta x \rightarrow 0} \Delta x \cdot \frac{1}{x^2(x + \Delta x)} = 0 \cdot \frac{1}{x^2(x + 0)} = 0$$

but plugging in zero is really not quite enough because it misses the point to the *approximation* we want. We want the (x, y) -graph of the function

$$F_{\Delta x}[x] = \frac{f[x + \Delta x] - f[x]}{\Delta x}$$

to approximate the graph of $f'[x]$ when Δx is small. Another way to say this is that we want the whole function $\varepsilon[x, \Delta x]$ to be small independent of x provided Δx is sufficiently small. Then we can move the microscope over these values of x and continue to see a straight line approximating $y = f'[x]$.

The factor Δx in $\Delta x \cdot \frac{1}{x^2(x+\Delta x)}$ does tend to zero; so, if we can bound the other factor, the whole function can be made small. The only way for $\frac{1}{x^2(x+\Delta x)}$ to get big is for x to be near zero. If we restrict x to stay away from the ‘bad’ point $x = 0$, then we can always make ε small. If $b > 0$ is fixed,

$$\varepsilon \approx 0 \quad \text{for all } |x| \geq b > 0 \quad \text{when } \delta x \approx 0$$

and the graphs of the functions $F_{\Delta x}[x]$ tend to the graph of $f'[x]$ for all $|x| \geq b$.

Increment of $f[x] = 1/x$:

$$\begin{aligned} \frac{1}{x + \Delta x} - \frac{1}{x} &= \frac{-1}{x^2} \cdot \Delta x + \varepsilon \cdot \Delta x \\ f[x + \Delta x] - f[x] &= f'[x] \cdot \Delta x + \varepsilon \cdot \Delta x \end{aligned}$$

with $f'[x] = -\frac{1}{x^2}$ and $\varepsilon = \Delta x \cdot \frac{1}{x^2(x + \Delta x)}$.

We summarize the knowledge that ε can be made small by making Δx small by writing

$$y = \frac{1}{x} \quad \Rightarrow \quad dy = \frac{-1}{x^2} dx$$

This notation means that under sufficient magnification the gap between the curve and its tangent will be appear small as shown in Figure 5.3 for $x = 3/2$. The formulas are not valid if $x = 0$.

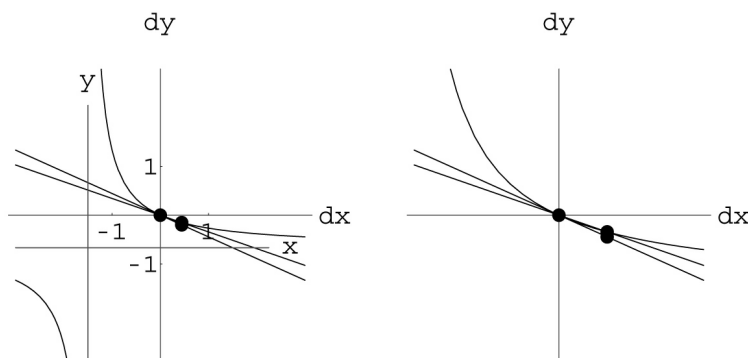


Figure 5.3: The gap near $x = 3/2$ on $y = 1/x$

Example 5.3 *Exceptional Numbers and the Derivative of $y = \sqrt{x}$*

Here is another example of the algebraic part of computing increments. We follow the steps of Procedure 5.1 and Exercise 5.1. (See Exercise 28.7.3 for help with these computations.)

1) Compute $\frac{f[x + \Delta x] - f[x]}{\Delta x}$

$$\begin{aligned} f[x + \Delta x] - f[x] &= \sqrt{x + \Delta x} - \sqrt{x} \\ &= \frac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \Delta x} + \sqrt{x})}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \cdot \Delta x \\ \frac{f[x + \Delta x] - f[x]}{\Delta x} &= \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \end{aligned}$$

2) Compute

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x}$$

Although there certainly may be difficulties near $x = 0$ (see Figure 5.4), if x is fixed and positive,

$$\lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x + 0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

3) Calculate the error gap

$$\varepsilon = \frac{f[x + \Delta x] - f[x]}{\Delta x} - f'(x)$$

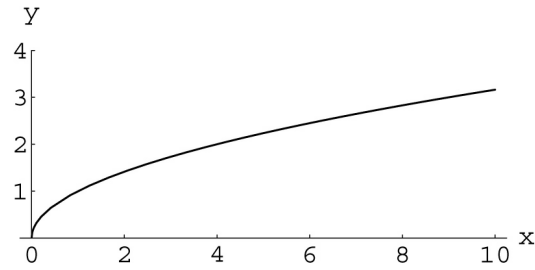
(see Exercise 28.7.3):

$$\begin{aligned} \varepsilon &= \frac{f[x + \Delta x] - f[x]}{\Delta x} - f'(x) \\ &= \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} - \frac{1}{2\sqrt{x}} \\ &= \frac{-1}{2\sqrt{x}(\sqrt{x + \Delta x} + \sqrt{x})^2} \cdot \Delta x \end{aligned}$$

4) Show that $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$. It seems “clear” that

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \varepsilon &= \lim_{\Delta x \rightarrow 0} \frac{-1}{2\sqrt{x}(\sqrt{x + \Delta x} + \sqrt{x})^2} \cdot \Delta x \\ &= \frac{-1}{2\sqrt{x}(\sqrt{x + 0} + \sqrt{x})^2} \cdot 0 \\ &= \frac{-1}{8x\sqrt{x}} \cdot 0 = 0 \end{aligned}$$

and this shortcut computation (plugging in 0) is justified as a function approximation for all x as long as the term $\frac{-1}{8x\sqrt{x}}$ cannot get large. This is guaranteed by making $x \geq b$ for some fixed positive $b > 0$.

Figure 5.4: $y = \sqrt{x}$

Increment of $f[x] = \sqrt{x}$:

$$f[x + \Delta x] - f[x] = f'[x] \cdot \Delta x + \varepsilon \cdot \Delta x$$

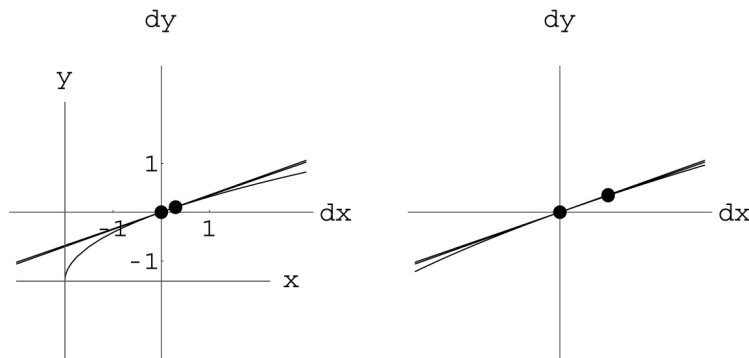
$$\sqrt{x + \Delta x} - \sqrt{x} = \frac{1}{2\sqrt{x}} \cdot \Delta x + \varepsilon \cdot \Delta x$$

with $f'[x] = \frac{1}{2\sqrt{x}}$ and $\varepsilon = -\Delta x / (2\sqrt{x}(\sqrt{x + \Delta x} + \sqrt{x})^2)$.

We summarize the knowledge that ε can be made small by making Δx small by writing

$$y = \sqrt{x} \quad \Rightarrow \quad dy = \frac{1}{2\sqrt{x}} dx$$

This notation means that under sufficient magnification the gap between the curve and its tangent will appear small as shown in Figure 5.5 for $x = 2/3$. These formulas are not valid if $x \leq 0$.

Figure 5.5: The gap near $x = 2/3$ on $y = \sqrt{x}$

Exercise Set 5.1

1. $y = x^p \Rightarrow dy = p x^{p-1} dx$, $p = 1, 2, 3, \dots$

For each $f[x] = x^p$ below:

- (a) Compute $\frac{f[x + \Delta x] - f[x]}{\Delta x}$ and simplify.
- (b) Compute $f'[x] = \lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x}$. Give an intuitive justification why your limit is correct. Does x need to be bounded, or can it vary arbitrarily as Δx tends to zero?
- (c) Use your limit $f'[x]$ to solve for ε and write the increment equation:

$$\begin{aligned} f[x + \Delta x] - f[x] &= f'[x] \cdot \Delta x + \varepsilon \cdot \Delta x \\ &= [\text{term in } x \text{ but not } \Delta x]\Delta x + [\text{observed microscopic error}]\Delta x \end{aligned}$$

Notice that we can solve the increment equation for $\varepsilon = \frac{f[x + \Delta x] - f[x]}{\Delta x} - f'[x]$

- (d) Show that $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$. “Show” this in any way that you consider reasonable. Does x need to be bounded, or can it vary arbitrarily as Δx tends to zero?
- i. If $f[x] = x^1$, then $f'[x] = 1x^0 = 1$ and $\varepsilon = 0$.
 - ii. If $f[x] = x^2$, then $f'[x] = 2x$ and $\varepsilon = \Delta x$.
 - iii. If $f[x] = x^3$, then $f'[x] = 3x^2$ and $\varepsilon = (3x + \Delta x)\Delta x$.
 - iv. If $f[x] = x^4$, then $f'[x] = 4x^3$ and $\varepsilon = (6x^2 + 4x\Delta x + \Delta x^2)\Delta x$.
 - v. If $f[x] = x^5$, then $f'[x] = 5x^4$ and $\varepsilon = (10x^3 + 10x^2\Delta x + 5x\Delta x^2 + \Delta x^3)\Delta x$.

If you have difficulty with this exercise, see the high school review Exercise powerquotients. Also see the program `SymbIncr` in this chapter’s folder. If you want practice computing limits, see the Math Background chapter on computing limits on the CD accompanying this book.

Problem 5.1 Use Procedure 5.1 to find $f'[x]$, and write the whole increment approximation

$$f[x + \Delta x] - f[x] = f'[x] \Delta x + \varepsilon \Delta x$$

showing that $\varepsilon \rightarrow 0$, when $\Delta x \rightarrow 0$.

1. (a) $f[x] = x^n$, n a positive integer
- (b) $f[x] = \frac{1}{x^2}$
- (c) $f[x] = \sqrt[3]{x}$

You can use the computer to help with your symbolic computations. See the program `SymbIncr` in this chapter’s folder and verify your predictions using the program `Micro1D`.

5.2 Moving the Microscope

This section uses interval notation to give a technical definition of local linearity. This “uniform” limit allows us to “move” the microscope.

A summary of the ε -gap computations so far is

$$\begin{aligned} y = f[x] = x^p &\Rightarrow dy = f'[x] dx = px^{p-1} dx, & p = 1, 2, 3, 4, 5 \\ y = f[x] = \frac{1}{x} &\Rightarrow dy = f'[x] dx = \frac{-1}{x^2} dx \\ y = f[x] = \frac{1}{x^2} &\Rightarrow dy = f'[x] dx = \frac{-2}{x^3} dx \\ y = f[x] = \sqrt{x} &\Rightarrow dy = f'[x] dx = \frac{1}{2\sqrt{x}} dx \\ y = f[x] = \sqrt[3]{x} &\Rightarrow dy = f'[x] dx = \frac{1}{3\sqrt[3]{x^2}} dx \end{aligned}$$

More than the summary, we know that the size of the gap (given by ε) viewed in a microscope of power $1/\Delta x$ goes to zero even as x varies, provided we avoid “bad” points. For the integer powers, we need to have x bounded, $|x| \leq b$ for a fixed b as you saw in Exercise 5.1.1. (“ $\pm\infty$ ” are the “bad” points in this case.) The other “bad” points are fairly obvious from the summary above. If $x = 0$, the function $1/x$ and its derivative $-1/x^2$ are undefined. We have to expect trouble there. If $x < 0$, the function \sqrt{x} is undefined, but even if $x = 0$, where \sqrt{x} is defined, the derived function $1/(2\sqrt{x})$ is undefined, so we expect trouble. All we need is a way to say where “good” approximations take place.

To give a general approach, we want to phrase the exceptions in terms of intervals.

Definition 5.1 Notation for Open and Compact Intervals

If a and b are numbers, we define open intervals as follows

$$\begin{aligned} (a, b) &= \{x : a < x < b\} \\ (-\infty, b) &= \{x : x < b\} \\ (a, \infty) &= \{x : a < x\} \end{aligned}$$

We define compact (or “closed and bounded”) intervals by $[a, b] = \{x : a \leq x \leq b\}$

The condition of “tangency” is expressed by the microscopic error formula given next.

Informal Smoothness:

The function $f'[x]$ is the derivative of the function $f[x]$ if whenever we make a small change $\delta x \approx 0$ in input x in the interval of differentiability (a, b) , then the change in output satisfies

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

with error $\varepsilon \approx 0$. This looks like

Informally, this approximation remains valid if we move the microscope anywhere inside an interval of “good” points. The approximation means that a microscopic view of a tiny piece of the graph $y = f[x]$ looks the same as the linear graph $dy = f'[x] \cdot dx$. (The lower case (small) Greek delta δ , indicates intuitively that when the difference in x is a sufficiently small amount, $\delta x \approx 0$, then the error $\varepsilon \approx 0$ is

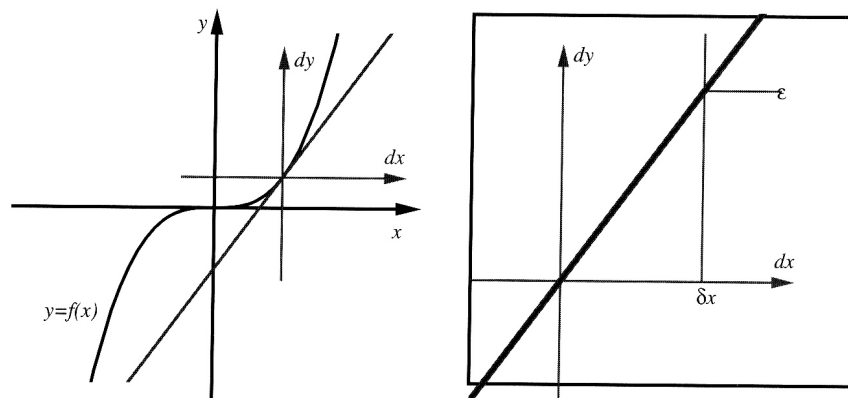


Figure 5.6: A symbolic microscope

small.) When we say $f'[x]$ is the derivative of $f[x]$ we mean that this local approximation is valid. We have shown this approximation directly for the functions summarized at the beginning of the section on appropriate compact intervals described in detail at the end of this section.

The rules of calculus are wonderful: They tell you where the trouble is going to occur.

Procedure 5.2 *The Graph of the Linear Function Given by*

$$dy = m dx$$

in local (dx, dy) -coordinates at $(x, f[x])$ is the tangent line to the explicit nonlinear graph

$$y = f[x]$$

provided $m = f'[x]$ and $f'[x]$ can be computed by the rules yielding a formula valid in an interval around x .

Specifically,

Theorem 5.1 *Successful Rules Imply Linear Approximation*

Suppose the derivative $\frac{dy}{dx} = f'[x]$ can be computed from an explicit formula $y = f[x]$ using the rules of Chapter 6. Also, suppose that both $f[x]$ and $f'[x]$ are defined on the compact interval $[\alpha, \beta]$. Then the size of the gap, ε in

$$f[x + \Delta x] - f[x] = f'[x] \cdot \Delta x + \varepsilon \cdot \Delta x$$

can be made small for all x in $[\alpha, \beta]$ by choosing a sufficiently small Δx . (For all x in $[\alpha, \beta]$ if $\Delta x = \delta x \approx 0$, then $\varepsilon \approx 0$.)

The complete technical definition of smoothness is given in the Mathematical Background materials on the CD accompanying this text. The background also gives a proof of Theorem 5.1. Here is the technical definition.

Definition 5.2 *Technical Smoothness (See Mathematical Background for Details)*

A real function $f[x]$ is called smooth (or differentiable or derivable) on the open interval (a, b) if there is a function $f'[x]$ such that for every compact subinterval $[\alpha, \beta]$ in (a, b) , the function limit

$$\lim_{\Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = f'[x] \quad \text{uniformly for all } x \text{ in } [\alpha, \beta]$$

In the Mathematical Background we show that the following are equivalent to this definition:

1. A real function $f[x]$ is smooth on the real interval (a, b) if there is another real function $f'[x]$, called the derivative of $f[x]$, such that whenever $a < x < b$ and x is a bounded hyperreal number and not infinitely near a or b , then an infinitesimal increment of the dependent variable is approximately linear, that is,

$$f[x + \delta x] - f[x] = f'[x] \delta x + \varepsilon \delta x$$

where the error ε is infinitesimal, whenever δx is infinitesimal.

2. A real function $f[x]$ is smooth on the open interval (a, b) if there is a function $f'[x]$ such that for every c in (a, b) , the double limit converges,

$$\lim_{x \rightarrow c, \Delta x \rightarrow 0} \frac{f[x + \Delta x] - f[x]}{\Delta x} = f'[c]$$

The Mathematical Background also has a chapter on computing uniform limits of formulas like the gaps ε we have found so far.

We do not have to worry about all the technical details, but we do want to understand the role of points where either $f[x]$ or $f'[x]$ is an undefined formula. The following examples are all smooth functions. By Theorem 5.1, the “proof” that they are smooth just amounts to valid use of basic rules, in this case the Power Rule.

Example 5.4 *Domains of Approximation for $y = x^p$, $p = 1, 2, 3, \dots$*

The functions $y = x^n$ and their derivatives $\frac{dy}{dx} = nx^{n-1}$ for $n = 1, 2, 3, \dots$ are defined for all real x in $(-\infty, \infty)$. Theorem 5.1 says they are differentiable on the open interval $(-\infty, \infty)$, and this means that for any pair of real numbers, $\alpha < \beta$, the gap ε can be made small over the compact interval $[\alpha, \beta]$ by choosing a single, small enough Δx . You cannot make ε small for the whole real line, $(-\infty, \infty)$.

Example 5.5 *Domains of Approximation for $y = x^p$, $p = -1, -2, \dots$*

The functions $y = \frac{1}{x^n}$ and their derivatives $\frac{dy}{dx} = -\frac{n}{x^{n+1}}$, for $n = 1, 2, 3, \dots$ are defined for all nonzero real x in $(-\infty, 0)$ or $(0, \infty)$, but not at $x = 0$. Theorem 5.1 says they are differentiable on the open intervals $(-\infty, 0)$ and $(0, \infty)$. This means that for pairs of real numbers, $\alpha < \beta < 0$ or $0 < \alpha < \beta$, the gap ε can be made small over the whole compact interval $[\alpha, \beta]$ by choosing a single, small enough Δx . You cannot make ε small for the whole interval, $(-\infty, 0)$ or $(0, \infty)$ or for $[\alpha, \beta]$ if $\alpha < 0 < \beta$.

Example 5.6 *Domains of Approximation for $y = x^p$, $p = 1/2, 1/3, \dots$*

The function $y = \sqrt{x}$ and its derivative $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ are defined for all positive real x in $(0, \infty)$. Both are NOT defined at $x = 0$. Theorem 5.1 says they are differentiable on the open interval $(0, \infty)$, so for pairs of real numbers $0 < \alpha < \beta$, the gap ε can be made small over the compact interval $[\alpha, \beta]$ by choosing a single, small enough Δx . You cannot make ε small for the whole interval $(0, \infty)$.

Exercise Set 5.2

You can see this function convergence for yourself on the computer.

1. ε on the Computer

Run the program *DfctLimit* to show graphically that the gap errors of all the following functions tend to zero AS FUNCTIONS OF x away from the “bad” points.

$$\begin{aligned} y = f[x] = x^p &\Rightarrow dy = f'[x] dx = px^{p-1} dx, \quad p = 1, 2, 3, 4, 5 \\ y = f[x] = \frac{1}{x} &\Rightarrow dy = f'[x] dx = \frac{-1}{x^2} dx \\ y = f[x] = \frac{1}{x^2} &\Rightarrow dy = f'[x] dx = \frac{-2}{x^3} dx \\ y = f[x] = \sqrt{x} &\Rightarrow dy = f'[x] dx = \frac{1}{2\sqrt{x}} dx \\ y = f[x] = \sqrt[3]{x} &\Rightarrow dy = f'[x] dx = \frac{1}{3\sqrt[3]{x^2}} dx \end{aligned}$$

2. A View in the Microscope

You are told that a certain function $y = f[x]$ has a derivative for all values of x , $f'[x]$. At the point $x = 1$, we know that $f'[1] = -2/3$. Sketch what you would see in a very powerful microscope focused on the graph $y = f[x]$ above the point $x = 1$.

Compare your work on the next problem about kinks with Exercise 3.2.4.

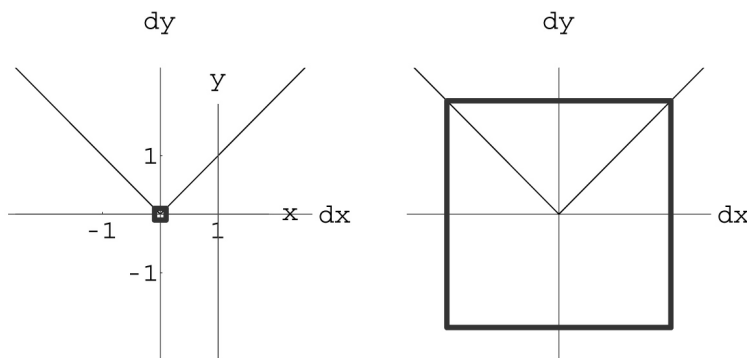
Problem 5.2 ε on the Computer and Analytically for a Kink Run the program *DfctLimit* on the function

$$y = f[x] = \sqrt{x^2 + 2x + 1}$$

which has derivative

$$\frac{dy}{dx} = f'[x] = \frac{x + 1}{\sqrt{x^2 + 2x + 1}}$$

Show that the function and its derivative are NOT BOTH defined at $x = -1$. Verify analytically that the gap you see on the computer does NOT tend to zero.

Figure 5.7: $y = \sqrt{x^2 + 2x + 1}$ near $x = -1$

5.3 Trigonometric Derivatives

The gaps ε for $y = f[x] = \text{Sin}[x]$, $y = f[x] = \text{Cos}[x]$, and $y = f[x] = \text{Tan}[x]$ are calculated in this section by comparing the length of a segment of the unit circle with the vertical and horizontal projections from the ends of the segment.

The derivative of sine in radians is cosine and the derivative of cosine in radians is -sine. These important facts can be seen by magnifying the unit circle. We assume that you know the definition of radian measure of angles and the associated fact that $(\text{Cos}[\theta], \text{Sin}[\theta])$ is the (x, y) -point on the unit circle at the angle θ , measured counterclockwise from the x -axis. (See Chapter 28, Section 5 and Figure 5.8.)

In this section, we use the informal version of Definition 5.2 and work from the relationship between the sine and cosine and the length along the unit circle shown on Figure 5.8. Consider what happens as we move from a point $(\text{Cos}[\theta], \text{Sin}[\theta])$ to a nearby point $(\text{Cos}[\theta + \delta\theta], \text{Sin}[\theta + \delta\theta])$. We magnify the unit circle, noting on Figure 5.9 that the more we magnify, the straighter the magnified portion of the circle appears.

The figure with small $\delta\theta \approx 0$ appears to be a triangle at magnification $1/\delta\theta$. The length of the hypotenuse of the apparent triangle is $\delta\theta$ because we use radian measure. (Degrees are not the distance along a unit circle.) The radii coming from the larger figure appear to meet it at right angles, so the apparent triangle is similar to the large triangle at the left with hypotenuse 1 and sides $\text{Sin}[\theta]$, $\text{Cos}[\theta]$. (You may have to do some geometry to convince yourself of this, since the corresponding edges are at right angles to one another.) The sides of the apparent triangle are the differences in sine and cosine, with cosine decreasing - hence a negative sign.

Figure 5.9 is the microscopic view of the circle that gives us the results

Consider the apparent similarity, comparing the long sides of the two triangles,

$$\frac{\text{Cos}[\theta]}{1} \approx \frac{\delta \text{sin}}{\delta\theta} = \frac{\text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta]}{\delta\theta}$$

Because we only know the apparent triangle up to a small error, we write only approximate similarity. To be explicit, let the difference equal,

$$\varepsilon = \frac{\text{Cos}[\theta]}{1} - \frac{\text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta]}{\delta\theta}$$

Now do a little algebra to see,

$$\text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta] = \text{Cos}[\theta] \cdot \delta\theta + \varepsilon \cdot \delta\theta$$

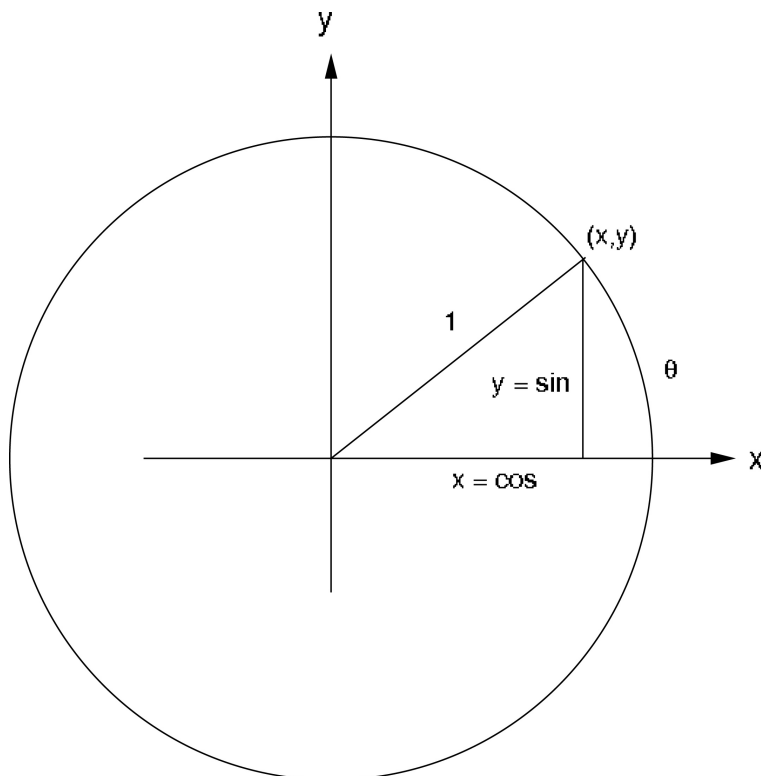


Figure 5.8: Sine and Cosine as a point on the Unit Circle

with $\varepsilon \approx 0$ whenever $\delta\theta \approx 0$. This has the form

$$f[\theta + \delta\theta] - f[\theta] = f'[\theta] + \varepsilon \cdot \delta\theta$$

with $f[\theta] = \text{Sin}[\theta]$, $f'[\theta] = \text{Cos}[\theta]$, and proves half of the following:

$$\begin{array}{ll} y = \text{Sin}[\theta] \Rightarrow \frac{dy}{d\theta} = \text{Cos}[\theta] & \text{or} \quad \text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta] = \text{Cos}[\theta] \delta\theta + \varepsilon \cdot \delta\theta \\ y = \text{Cos}[\theta] \Rightarrow \frac{dy}{d\theta} = -\text{Sin}[\theta] & \text{Cos}[\theta + \delta\theta] - \text{Cos}[\theta] = -\text{Sin}[\theta] \delta\theta + \varepsilon \cdot \delta\theta \end{array}$$

For θ in radians, with $\varepsilon \approx 0$ if $\delta\theta \approx 0$ (θ can take any value.)

Example 5.7 *Increments of Sine*

The most important meaning of the increment formula for Sin is simply that a small piece of the graph is given by the linear equation with slope $m = \text{Cos}[x]$. For example, suppose that we want to estimate the sine of 29 degrees. We know sine of 30 degrees and we can take the increment of -1 degree, using the “microscope equation.” We must first convert to radian measure because the increment formulas above are

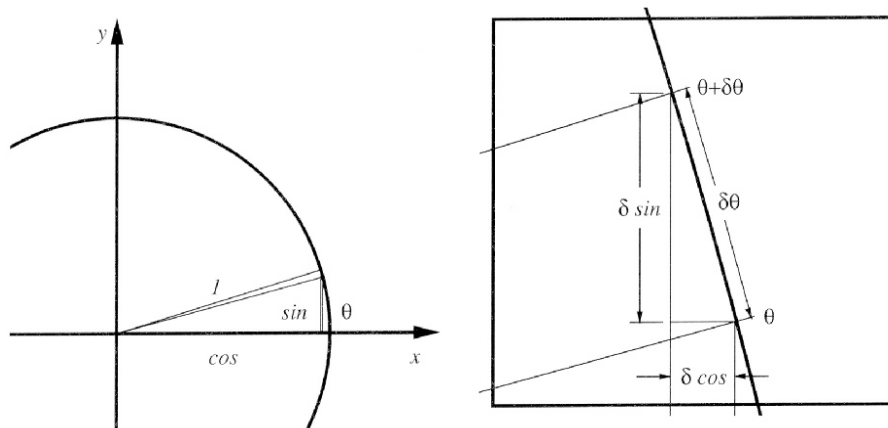


Figure 5.9: Derivatives of sine and cosine

valid only in radian measure. We take $\theta = \frac{\pi}{6}$ and $\delta\theta = -\frac{\pi}{180} \approx 0$,

$$\begin{aligned}
 f[x + \delta x] - f[x] &= f'[x] \cdot \delta x + \varepsilon \cdot \delta x \\
 f[\theta + \delta\theta] - f[\theta] &= f'[\theta] + \varepsilon \cdot \delta\theta \\
 \text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta] &= \text{Cos}[\theta] \cdot \delta\theta + \varepsilon \cdot \delta\theta \\
 \text{Sin}\left[\frac{\pi}{6} - \frac{\pi}{180}\right] - \frac{1}{2} &= \frac{\sqrt{3}}{2} \cdot \frac{-\pi}{180} + \varepsilon \cdot \delta\theta \\
 \text{Sin}\left[\frac{\pi}{6} - \frac{\pi}{180}\right] &\approx \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{-\pi}{180} \\
 \text{Sin}\left[\frac{\pi}{6} - \frac{\pi}{180}\right] &\approx 0.484885
 \end{aligned}$$

The computer's approximation of sine of 29 degrees is 0.48481.

Example 5.8 Limits of Sine

The previous example can be cast in limit notation as: Find

$$\lim_{\Delta\theta \rightarrow 0} \frac{\text{Sin}\left[\frac{\pi}{6} + \Delta\theta\right] - \text{Sin}\left[\frac{\pi}{6}\right]}{\Delta\theta}$$

The solution is to recognize this as a special case of the limit defining the derivative,

$$\lim_{\Delta\theta \rightarrow 0} \frac{\text{Sin}[\theta + \Delta\theta] - \text{Sin}[\theta]}{\Delta\theta} = \text{Cos}[\theta]$$

with $\theta = \pi/6$, or to use the increment approximation,

$$\begin{aligned}
 \text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta] &= \text{Cos}[\theta] \cdot \delta\theta + \varepsilon \cdot \delta\theta \\
 \frac{\text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta]}{\delta\theta} &= \text{Cos}[\theta] + \varepsilon \\
 \frac{\text{Sin}\left[\frac{\pi}{6} + \delta\theta\right] - \text{Sin}\left[\frac{\pi}{6}\right]}{\delta\theta} &= \text{Cos}\left[\frac{\pi}{6}\right] + \varepsilon
 \end{aligned}$$

and recall that

$$\lim_{\Delta\theta \rightarrow 0} \frac{\text{Sin}[\frac{\pi}{6} + \Delta\theta] - \text{Sin}[\frac{\pi}{6}]}{\Delta\theta} = L$$

is the number L the expression approximates when $\Delta\theta = \delta\theta \approx 0$ is small,

$$\frac{\text{Sin}[\frac{\pi}{6} + \delta\theta] - \text{Sin}[\frac{\pi}{6}]}{\delta\theta} \approx \text{Cos}[\frac{\pi}{6}]$$

5.3.1 Differential Equations and Functional Equations

It is intuitively clear that magnified circles appear straighter and straighter, but complete justification of the local linearity of sine and cosine requires that we really show that the magnified increment of the circle is close to a triangle. We will not do this here except to make two specific uses of identities that are important in their own right. More details are contained in the Mathematical Background chapter on Functional Identities.

The formula

$$(\text{Sin}[\theta])^2 + (\text{Cos}[\theta])^2 = 1$$

simply says that sine and cosine lie on the unit circle. If $x = \text{Cos}[\theta]$ and $y = \text{Sin}[\theta]$, $x^2 + y^2 = 1$ is the equation of the unit circle.

Rather than using the increment approximation based on a greatly magnified circle, we could use the exact addition formulas to obtain increments of trig functions. In the case of the sine,

$$\text{Sin}[\theta + \delta\theta] = \text{Sin}[\theta] \text{Cos}[\delta\theta] + \text{Cos}[\theta] \text{Sin}[\delta\theta]$$

For example,

$$\text{Sin}[\pi/6 + \delta\theta] = \frac{1}{2} \text{Cos}[\delta\theta] + \frac{\sqrt{3}}{2} \text{Sin}[\delta\theta]$$

These are exact formulas for the increments, but we need to obtain the differential approximations

$$\text{Sin}[\delta\theta] = \delta\theta + \varepsilon_1 \cdot \delta\theta$$

$$\text{Cos}[\delta\theta] = 1 + \varepsilon_2 \cdot \delta\theta$$

to complete the last step in proving the local linear approximation.

The point we wish to illustrate is this:

The differential

$$d(\text{Sin}[\theta]) = \text{Cos}[\theta] d\theta$$

is a sort of simplified version of the functional identity

$$\begin{aligned} \text{Sin}[\theta + \delta\theta] - \text{Sin}[\theta] &= \text{Sin}[\theta] \text{Cos}[\delta\theta] - \text{Sin}[\theta] + \text{Cos}[\theta] \text{Sin}[\delta\theta] \\ &= \text{Cos}[\theta] \delta\theta + \text{Cos}[\theta] (\text{Sin}[\delta\theta] - \delta\theta) + \text{Sin}[\theta] (\text{Cos}[\delta\theta] - 1) \\ &= \text{Cos}[\theta] \delta\theta + \varepsilon \cdot \delta\theta \end{aligned}$$

that discards the error term ε .

We know $\varepsilon \approx 0$ because magnified circles appear straighter and straighter as the magnification increases. This observation gives us two interesting limits. Since

$$\varepsilon = \text{Cos}[\theta] \frac{\text{Sin}[\delta\theta] - \delta\theta}{\delta\theta} + \text{Sin}[\theta] \frac{\text{Cos}[\delta\theta] - 1}{\delta\theta}$$

and since this is small for all θ , we have $\frac{\text{Sin}[\delta\theta]-\delta\theta}{\delta\theta} = \frac{\text{Sin}[\delta\theta]}{\delta\theta} - 1 \approx 0$ and $\frac{(\text{Cos}[\delta\theta]-1)}{\delta\theta} \approx 0$, or

$$\lim_{\Delta\theta \rightarrow 0} \frac{\text{Sin}[\Delta\theta]}{\Delta\theta} = 1 \quad \& \quad \lim_{\Delta\theta \rightarrow 0} \frac{\text{Cos}[\Delta\theta] - 1}{\Delta\theta} = 0$$

These limits are just the derivatives of sine and cosine at zero.

Exercise Set 5.3

1. A View in the Microscope

- (a) Sketch the view you would see in a powerful microscope focused on the graph $y = \text{Sin}[\theta]$ above the point where $\theta = \pi/3$. Verify your prediction using the program **Micro1D**.

2. Derivative of Cosine

- (a) Use a powerful microscope to prove that the differential of cosine is minus sine,

$$y = \text{Cos}[\theta] \quad \Rightarrow \quad dy = -\text{Sin}[\theta] d\theta$$

- (b) Write an exact formula for $\text{Cos}[x+\delta x] = \text{Cos}[x] \text{Cos}[\delta x] \dots$ using the addition formula for cosine. Compare the exact formula with the increment approximation obtained from the microscopic view of the circle. What is the exact formula for ε ?
- (c) Approximate the value of cosine of 46 degrees using the linear increment approximation, discarding the $\varepsilon \delta x$ term. Give your answer in terms of exact constants such as π and $\sqrt{2}$ as well as numerically. Compare your approximation with the computer or your scientific calculator's approximation.
- (d) Use the increment approximation for cosine to show that

$$\lim_{\Delta\theta \rightarrow 0} \frac{\text{Cos}[\frac{\pi}{4} + \Delta\theta] - \text{Cos}[\frac{\pi}{4}]}{\Delta\theta} = \frac{1}{\sqrt{2}}$$

3. ε on the Computer

- (a) Run the program **DfctLimit** to show graphically that the gap errors of $y = \text{Sin}[x]$, $y = \text{Cos}[x]$, and $y = \text{Tan}[x]$ tend to zero AS FUNCTIONS OF x for appropriate compact intervals $[\alpha, \beta]$. (HINT: Where are the functions and their derivatives defined?)
-

The point of the next problem is that we can compute the increments of the tangent function directly. Later you will be able to differentiate $\text{Tan}[\theta]$ with rules from Chapter 6.

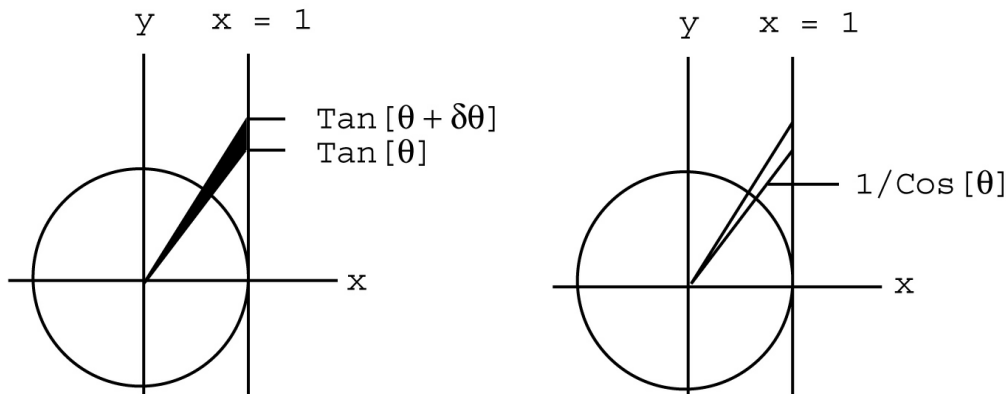


Figure 5.10: An increment of tangent and the secant Function

Problem 5.3 DERIVATIVE OF TANGENT (*Optional*)

Find the differential of the tangent function by examining an increment in the figures below. The segment on the line $x = 1$ between two rays from the circle is the increment of the tangent, because SOH-CAH-TOA with adjacent side of length 1 gives $\text{Tan}[\theta]$ as the length of the segment on $x = 1$ between the x -axis and the ray.

The area of the triangle in Figure 5.10 with $\delta y = \delta(\text{Tan}[\theta])$ as its tiny vertical side is $\frac{1}{2}\delta \text{Tan}[\theta]$, because the “height” is 1 for the “base” of $\delta \text{Tan}[\theta]$.

The length of the ray at θ out to $x = 1$ is $1/\text{Cos}[\theta]$. Why?

Show the lower estimate

$$\frac{1}{2} \frac{1}{\text{Cos}^2[\theta]} \delta\theta \leq \frac{1}{2} \delta \text{Tan}[\theta]$$

by finding the area of the circular sector at the left in Figure 5.11. (Note that the area of a circular sector of radius r and angle ϕ is $\frac{1}{2} r^2 \cdot \phi$.)

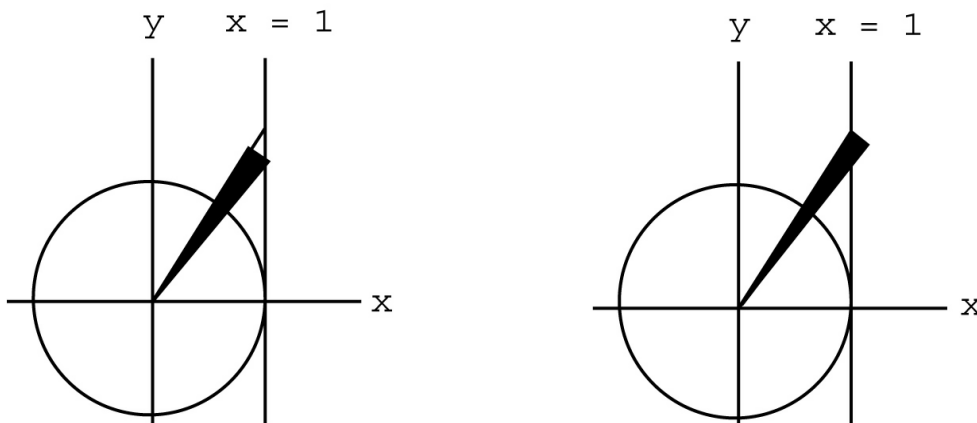


Figure 5.11: A lower estimate and an upper estimate

Similarly, show

$$\frac{1}{\text{Cos}^2[\theta]} \delta\theta \leq \delta \text{Tan}[\theta] \leq \frac{1}{\text{Cos}^2[\theta + \delta\theta]} \delta\theta$$

(What is the radius of the sector on the right in Figure 5.11?)

The area of both of the sectors in Figure 5.11 is $\frac{\delta\theta}{2\text{Cos}^2[\theta]} + \varepsilon \cdot \delta\theta$, with $\varepsilon \approx 0$ when $\delta\theta \approx 0$. Why? When is $\frac{1}{\text{Cos}[\theta]} \approx \frac{1}{\text{Cos}[\theta + \delta\theta]}$ for $\delta\theta \approx 0$?
 Prove that

$$y = \text{Tan}[\theta] \quad \Rightarrow \quad dy = \frac{1}{\text{Cos}^2[\theta]} d\theta$$

provided $\text{Cos}[\theta] \neq 0$.

5.4 Derivatives of Log and Exp

The gaps ε for $y = e^x$ and $y = \text{Log}[x]$ are discussed in this section.

The important functional identities of exponential functions are as follows:

Theorem 5.2 Laws of Exponents

For a positive base $a > 0$ and any real numbers p and q

$$\begin{aligned} a^{-p} &= \frac{1}{a^p} & a^p \cdot a^q &= a^{p+q} \\ a^{1/p} &= \sqrt[p]{a} & (a^p)^q &= a^{p \cdot q} \end{aligned}$$

We want to use these properties to show what we need to estimate in order to differentiate log and exponential functions. (Practice with the rules can be found in Chapter 28, Section 4, if your skills are rusty.)

Example 5.9 The Exact Increment of $y = a^x$

We write an exact formula for the difference $a^{x+\Delta x} - a^x$ in terms of a^x ,

$$\begin{aligned} a^{x+\Delta x} - a^x &= a^x \cdot a^{\Delta x} - a^x \\ &= a^x (a^{\Delta x} - 1) \\ \frac{a^{x+\Delta x} - a^x}{\Delta x} &= a^x \frac{a^{\Delta x} - 1}{\Delta x} \end{aligned}$$

Notice that the last formula says

The rate of change of $y = a^x$ for a fixed change Δx beginning at x is proportional to a^x ,

$$\frac{a^{x+\Delta x} - a^x}{\Delta x} = \frac{a^{\Delta x} - 1}{\Delta x} a^x$$

rate of change = $K \cdot a^x$

where the constant of proportionality $K = K[\Delta x] = (a^{\Delta x} - 1)/\Delta x$ depends only on the change Δx , not x .

Example 5.10 *Constant Rate of Change for Linear Functions*

Suppose an unknown function $f[x]$ increases by a constant amount k every time x increases by another constant amount h . What sort of function is $f[x]$? The statement that a constant change of h in input makes a constant change of k in output is

$$f[x + h] - f[x] = k$$

or the rate of change is constant

$$\frac{f[x + h] - f[x]}{h} = m$$

with $m = k/h$. It is easy to see that the linear function

$$f[x] = m \cdot x + b$$

with $m = k/h$ has the needed property (it will satisfy the rate equation for every h .)

Linear functions change at a constant rate. Exponential functions change by a constant percentage for a constant change in input.

Example 5.11 *Constant PERCENTAGE Change for Exponential Functions*

Suppose an unknown function $f[x]$ increases by a constant percentage every time x increases by a constant h . For example, suppose $f[x]$ increases by a third, 33.3%, every time x increases by $1/2$. The change in f is

$$f\left[x + \frac{1}{2}\right] - f[x] = 33.3\% \text{ of } f[x]$$

$$f\left[x + \frac{1}{2}\right] - f[x] = \frac{1}{3}f[x]$$

We try to find an exponential solution, $f[x] = a^x$, of this functional equation

$$f\left[x + \frac{1}{2}\right] - f[x] = \frac{1}{3}f[x]$$

$$a^{x+\frac{1}{2}} - a^x = \frac{1}{3}a^x$$

$$a^x a^{\frac{1}{2}} - a^x = \frac{1}{3}a^x$$

$$a^{\frac{1}{2}} - 1 = \frac{1}{3}$$

$$a^{\frac{1}{2}} = \frac{4}{3}$$

$$a = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$$

The function $f[x] = \left(\frac{16}{9}\right)^x$ increases by one third every time x increases by one half.

Percentage Rate of Change as $\Delta x \rightarrow 0$

When Δx gets smaller and smaller, we would like to show that $K[\Delta x]$ converges to a constant $k_a = \lim_{\Delta x \rightarrow 0} K[\Delta x]$. In other words, when Δx is small, $\delta x \approx 0$, we would like to find an expression for the difference quotient of exponentials of the form,

$$\frac{a^{x+\delta x} - a^x}{\delta x} = a^x \frac{a^{\delta x} - 1}{\delta x} = a^x \cdot (k_a + \iota)$$

where the real constant k_a depends only on a and

$$k_a \approx \frac{a^{\delta x} - 1}{\delta x} \quad \text{or} \quad \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x} = k_a$$

It turns out that the mysterious constant k_a is $\text{Log}[a]$ (the natural logarithm) and $\iota \approx 0$, but this approximation is difficult to establish directly.

Notice that, if the limit converges, the result says the following:

The instantaneous rate of change of $y = a^x$ at x is proportional to a^x ,

$$\frac{d(a^x)}{dx} = k a^x$$

Moreover, the convergence is uniform for bounded x :

$$a^{x+\delta x} - a^x = k_a \cdot a^x + (\iota a^x) \cdot \delta x$$

The gap $\varepsilon = (\iota a^x) \approx 0$ for all bounded x .

The “natural” base $e \approx 2.71828 \dots$ plays an important role in solving the problem because the number $e \approx 2.71828$ is the unique number that makes

$$\frac{e^{\delta x} - 1}{\delta x} \approx 1 = k_e$$

for $\delta x \approx 0$, that is, $k_a = 1$ when $a = e$.

A mathematically simpler approach is to take the definition:

The function $y = \text{Exp}[x]$ is officially defined to be the unique solution to

$$\begin{aligned} y[0] &= 1 \\ \frac{dy}{dx} &= y \end{aligned}$$

This “definition” is based on two good guesses. First, that the derivative of an exponential is proportional to the quantity. We saw substantial evidence for this above. Second, that some special number e makes the constant of proportionality equal to 1. The Project on Direct Computation of the Derivative of Exponentials shows you more details on this guess and gives you a way to compute $e \approx 2.71828 \dots$

Technically, the approach relies on convergence of Euler’s approximation to differential equations. This is easier than the convergence problems in the direct approach to the exponential above and has many other applications. (We have not proved that Euler’s approximation converges, but we have seen it work in several examples: S-I-R, the canary, and so forth. The proof is in the Mathematical Background CD.)

Once we have made this the “official” definition, we can use Euler’s approximation to obtain the specific approximation

$$(1 + \delta x)^{x/\delta x} \approx e^x$$

We postpone further discussion to Chapter 8 but give the derivatives now. You may use these results as needed (without proof.)

Theorem 5.3 *Derivatives of Logs and Exponentials*

The derivative of the natural base exponential function is

$$y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = y$$

or, written in terms of the independent variable,

$$\frac{de^x}{dx} = e^x$$

The derivative of the natural base logarithm is

$$\frac{d\text{Log}[x]}{dx} = \frac{1}{x}$$

Once we know the derivative of the natural exponential and rules of differentiation, we can find the differentials of all exponentials. For this reason, the natural log and exponential play a major role in science and mathematics. Just as radian measure makes the calculus of trig functions “natural,” the $e \approx 2.71828 \dots$ base for logs and exponentials makes their calculus “natural.”

Exercise Set 5.4

Fixed Percentage Changes

1. Find an exponential function $f[x] = a^x$ that doubles every time x increases by 1. Write this English question as a mathematical equation and solve it using properties of exponents.
2. Find an exponential function $f[x] = b^x$ that triples every time x increases by 1. Write this English question as a mathematical equation and solve it using properties of exponents.
3. Find an exponential function $f[x] = c^x$ that increases by 50% every time x increases by $1/2$ x -unit. Write this English question as a mathematical equation and solve it using properties of exponents and the logarithm.
4. Show that the exponential base a from the first part is NOT equal to c from the third part. Should not a function that increases by 50% in $1/2$ unit be the same as one that increases by 100% in 1 unit? Why not?
5. Let $f[x]$ be an unknown function, h and k unknown constants. Write the statement, “The change in $f[\cdot]$ as x increases by h equals k times $f[x]$.” as a mathematical equation. In other words, your equation should say, “ $f[x]$ increases by $k \times 100\%$ as x increases by h .”

A Doubling Exponential

6. Suppose algae cells in a warm pond double every 6 hours and at time $t = 0$ (hrs) there is one cell.
- How many cells are there in 6 hours? How many cells are there in 12 hours? How many in 18 hours?
 - How many 6-hour periods are there in t hours? (A formula.)
 - Give the number of cells n as a function of t ,

$$n[t] = 2^{t/6}$$

7. Suppose at time t_1 there are a billion cells.
- How many are there at time $t_1 + 6$?
 - What is the formula for the rate of growth of algae cells in a 6-hour period beginning at an unknown time t ? (Compare your work to Problem *algaeProb* and the program **ExpGth** of Chapter 28.)

The next exercise has you practice using the functional identities for the logarithm. The point of the exercise is that we need only one limit, $\eta \rightarrow 0$, and the functional identity.

The Derivative of Natural Log

8. Given that $\frac{d \text{Log}[x]}{dx} = \frac{1}{x}$, write the increment approximation for Log at $x = 1$ to show that

$$\text{Log}[1 + \delta x] = \delta x + \varepsilon \cdot \delta x$$

with $\varepsilon \approx 0$ when $\delta x \approx 0$.

9. Use properties of logs to show that

$$\frac{\text{Log}[x + \delta x] - \text{Log}[x]}{\delta x} = \frac{\text{Log}[1 + \frac{\delta x}{x}]}{\delta x}$$

10. Suppose you are given that

$$\text{Log}[1 + \eta] = \eta + \varepsilon \cdot \eta$$

with $\varepsilon \approx 0$ when $\eta \approx 0$. Use these two facts to prove that for all positive x , bounded away from 0,

$$\text{Log}[x + \delta x] = \text{Log}[x] + \frac{\delta x}{x} + \varepsilon \cdot \delta x$$

with $\varepsilon \approx 0$ when $\delta x \approx 0$.

11. Use the above to show that

$$\lim_{\Delta x \rightarrow 0} \frac{\text{Log}[x + \Delta x] - \text{Log}[x]}{\Delta x} = \frac{1}{x}$$

12. What is wrong with the following computation?

$$\frac{\text{Log}[x + \delta x] - \text{Log}[x]}{\delta x} = \frac{\text{Log}[x + \delta x - x]}{\delta x} = \frac{\text{Log}[\delta x]}{\delta x}$$

Percent Growth and the Natural Base

13. We know that exponentials grow at a constant *PERCENTAGE* rate for fixed steps. (See the program **PercentGth**.) Let $f[x] = e^{rx}$ and substitute this into the expression

$$\frac{1}{f[x]} \frac{f[x + \delta x] - f[x]}{\delta x} = k, \quad k \text{ constant}$$

14. Suppose you know that $\text{Log}[1 + \eta] = \eta + \varepsilon \cdot \eta$ with $\varepsilon \approx 0$ when $\eta \approx 0$. Show that $r \approx k$ when $\delta x \approx 0$.

ε on the Computer

15. Run the program **DfctLimit** to show graphically that the gap errors of $y = \text{Exp}[x]$ and $y = \text{Log}[x]$ tend to zero AS FUNCTIONS OF x for appropriate compact intervals $[\alpha, \beta]$. (HINT: Where are the functions and their derivatives defined?)

The next exercise has some practice using these derivatives in the increment approximation. Do not use your calculator until you have written the symbolic expressions. (You do not have to use it at all, but you can check your work if you wish.)

Natural Increments

16. Use the formula for the derivative of the natural exponential to write the increment approximation for $y = e^x$,

$$e^{x+\delta x} = e^x + [?] \cdot \delta x + \varepsilon \cdot \delta x$$

17. Use the formula for the derivative of the natural logarithm to estimate $\text{Log}[2.8] = \text{Log}[e + (2.8 - 2.71828 \dots)]$.

5.5 Continuity and the Derivative

This section shows that locally linear implies continuous and uniform derivatives are continuous.

We saw in the Exercise 3.2.1 that a function can be continuous but still not smooth or differentiable. An official definition of continuity is the following

Definition 5.3 *Continuity of $f[x]$*

A real function $f[x]$ is continuous at the real point a if $f[a]$ is defined and

$$\lim_{x \rightarrow a} f[x] = f[a]$$

Intuitively, this just means that $f[x]$ is close to $f[a]$ when x is close to a , for every $x \approx a$, $f[x]$ is defined and

$$f[x] \approx f[a]$$

Theorem 5.4 *Continuity of $f[x]$ and $f'[x]$*

If $f[x]$ is smooth on the interval $a < x < b$, then both $f[x]$ and $f'[x]$ are continuous at every point c in (a, b) .

INTUITIVE PROOF FOR $f[x]$:

Proof of continuity of f is easy algebraically but is obvious geometrically: A graph that is indistinguishable from linear clearly only moves a small amount in a small x -step. Algebraically, we want to show that if $x_1 \approx x_2$ then $f[x_1] \approx f[x_2]$. Take $x = x_1$ and $\delta x = x_2 - x_1$ and use the approximation $f[x_2] = f[x + \delta x] = f[x_1] + f'[x_1]\delta x + \varepsilon \delta x$ where $[f'[x_1] + \varepsilon]\delta x$ is medium times small = small, so $f[x_1] \approx f[x_2]$. That is the algebraic proof. Draw the picture on a small scale.

INTUITIVE PROOF FOR $f'[x]$:

Proof of continuity of $f'[x]$ requires us to view the increment from both ends. First take $x = x_1$ and $\delta x = x_2 - x_1$ and use the approximation

$$f[x_2] = f[x + \Delta x] = f[x_1] + f'[x_1]\delta x + \varepsilon_1\delta x.$$

Next let $x = x_2$, $\delta x = x_1 - x_2$ and use the approximation

$$f[x_1] = f[x + \Delta x] = f[x_2] + f'[x_2]\delta x + \varepsilon_2\delta x.$$

The different x -increments are negatives, so we have

$$f[x_1] - f[x_2] = f'[x_2](x_1 - x_2) + \varepsilon_2(x_1 - x_2)$$

and

$$f[x_2] - f[x_1] = f'[x_1](x_2 - x_1) + \varepsilon_1(x_2 - x_1)$$

Adding, we obtain

$$0 = \{ [f'[x_2] - f'[x_1]] + (\varepsilon_2 - \varepsilon_1) \} (x_1 - x_2)$$

Dividing by the non-zero $(x_1 - x_2)$, we see that

$$f'[x_2] = f'[x_1] + (\varepsilon_1 - \varepsilon_2), \quad \text{so} \quad f'[x_2] \approx f'[x_1]$$

NOTE:

The derivative defined in many calculus books is a weaker pointwise notion than the notion of smoothness we have defined. The weak derivative function need not be continuous. (The same approximation does not apply at both ends with the weak definition.) This is explained in the Mathematical Background Chapter on “Epsilon - Delta” Approximations.

Exercise Set 5.5

1. Suppose that $f[x]$ is smooth on an interval around a so that the “microscope” increment equation is valid. Suppose that $x \approx a$ so that $x = a + \delta x$ for $\delta x \approx 0$. Show that $f[x] = f[a + \delta x] \approx f[a]$; in other words, show that smooth real functions are continuous at real points.
2. Consider the real function $f[x] = 1/x$, which is undefined at $x = 0$. We could extend the definition by simply assigning $f[0] = 0$. Show that this function is not continuous at $x = 0$ but is continuous at every other real x .

3. Give an intuitive graphical description of the definition of continuity in terms of powerful microscopes and explain why it follows that smooth functions must be continuous.
 4. The function $f[x] = \sqrt{x}$ is defined for $x \geq 0$; there is nothing wrong with $f[0]$. However, our increment computation for \sqrt{x} above was not valid at $x = 0$ because a microscopic view of the graph focused at $x = 0$ looks like a vertical ray (or half-line). Explain why this is so, but show that $f[x]$ is still continuous “from the right;” that is, if $0 < x \approx 0$, then $\sqrt{x} \approx 0$ but $\frac{\sqrt{x}}{x}$ is very large.
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5.6 Projects and Theory

5.6.1 Hubble’s Law and the Increment Equation

$$R[t + \delta t] = R[t] + R'[t] \delta t + \varepsilon \delta t$$

Evidence of an expanding universe is one of the most important astronomical observations of this century. Light received from a distant galaxy is “old” light, generated millions of years ago at a time t_e when it was emitted. When this old light is compared to light generated at the time received t_r , it is found that the characteristic colors, or spectral lines, do not have the same wavelengths. All the current wavelengths are longer, $\lambda[t_e] < \lambda[t_r]$. This means that light is “redder” now; this is the famous red shift.

The Scientific Project on Hubble’s Law shows you an explanation for the expanding universe that is based just on using the increment approximation. Recently, there has been some reexamination of Hubble’s Law indicating that Hubble’s “constant” may not be constant. This is still compatible with an increment derivation of the law, which relies on the tiny time increment of only a few human generations.

5.6.2 Numerical Approximation of Exponential Derivatives

The project on Exponential Derivatives has you calculate the constants

$$k_a = \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

and then adjust a until you make $k_a = 1$. This is one way to compute the natural base $e \approx 2.71828\dots$

5.6.3 Small Enough Real Numbers or “Epsilons and Deltas”

The increment approximation used to estimate $\text{Sin}[“29 degrees”]$ was very close, but how do we know that the increment approximation gets close for real increments, not just close for small increments? The Mathematical Background chapter on “epsilons and deltas” answers this question. All the theorems are proved in detail in the Mathematical Background.