Chapter 8
The Natural Log and Exponential

This chapter treats the basic theory of logs and exponentials. It can be studied any time after Chapter 6. You might skip it now, but should return to it when needed.

The "natural" base exponential function and its inverse, the natural base logarithm, are two of the most important functions in mathematics. This is reflected by the fact that the computer has built-in algorithms and separate names for them:

$$y = e^x = \mathrm{Exp}[x] \qquad \Leftrightarrow \qquad x = \mathrm{Log}[y]$$

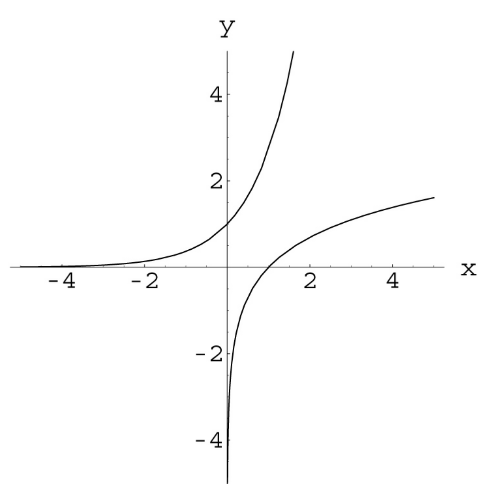


Figure 8.0:1: $y = \mathrm{Exp}\,[x]$ and $y = \mathrm{Log}\,[x]$

We did not prove the formulas for the derivatives of logs or exponentials in Chapter 5. This chapter defines the exponential to be the function whose derivative equals itself. No matter where we begin in terms of a basic definition, this is an essential fact. It is so essential that everything else follows from it. We call this the "official theory."

We already know the differentiation rules for log and exponential, and the basic high school review material about logs and exponentials is contained in Chapter 28. The main facts to memorize are

$$\begin{array}{lcl} \frac{de^t}{dt} = e^t & & \frac{d\,\mathrm{Log}[s]}{ds} = \frac{1}{s} \\ e^a \times e^b = e^{a+b} & & \mathrm{Log}[a \times b] = \mathrm{Log}[a] + \mathrm{Log}[b] \\ & \& & \\ (e^c)^t = e^{c \cdot t} & & \mathrm{Log}[a^p] = p \times \mathrm{Log}[a] \\ \mathrm{Log}[e^t] = t & & e^{\mathrm{Log}[s]} = s, \quad s > 0 \end{array}$$

Some of the graphical properties of these functions are formulated as limits, comparing them to power functions later in the chapter. Section 8.3 explains these "orders of infinity" more technically and shows how to build more limits from them. The basic limits say

---

- e^t goes to infinity faster than any power as $t \to \infty$.
- $\mathrm{Log}[t]$ tends to infinity slower than any root as $t \to \infty$.
- e^{-t} is positive but tends to zero faster than any reciprocal power as $t \to \infty$.
- $\mathrm{Log}[s] \downarrow -\infty$ as $s \downarrow 0$.

See the the computer programs **ExpGth** and **LogGth** in Chapter 28 for an intuitive explanation of these limits.

8.1 The Official Natural Exponential

One of the most important ways that exponential functions arise in science and mathematics is as the solution to linear growth and decay laws.

The differential equation $\dfrac{dy}{dt} = k\,y$ with a positive constant k represents proportional growth and with a negative constant represents proportional decay. We have already seen a decay law of this form in Newton's Law of Cooling or the Cool Canary Problem 4.1 and a growth law of this form in the first (false) conjecture of Galileo on the Law of Gravity, Problem 4.2. These laws simply say, "The rate of change of a quantity is proportional to the amount present." In Exercise 4.2.1, we solved the differential equation numerically, but now we will be able to solve these problems symbolically (exactly) in Problems 8.1 and 4.5.

The most noteworthy thing about the formulas in this chapter is this: The dependent variable y appears on both sides of the equation.
SOMETHING NEW:

$$\frac{dy}{dt} = k\,y$$

This is an important differential equation, not just another differentiation formula like the ones in Chapter 6. (Those equations all have explicit functions of the independent variable t on the right hand side, for example, $y = 3\,t^5 \Rightarrow \frac{dy}{dt} = 15\,t^4$.) It might be worthwhile to contrast this situation with the "growth form" of a linear function:

Theorem 8.1 *The Differential Equation of a Linear Function*

For appropriate constants k and Y_0, the following are equivalent:

$$\frac{dy}{dx} = k \quad \Leftrightarrow \quad y = k\,x + Y_0$$

The rate of change of y with respect to x is constant if and only if y varies linearly.

Theorem 8.2 *The Differential Equation of an Exponential Function*

For appropriate constants k and Y_0, the following are equivalent:

$$\frac{1}{y}\,\frac{dy}{dx} = k \quad \Leftrightarrow \quad y = Y_0\ e^{k\,x}$$

The rate of change of y is a constant percentage if and only if y varies exponentially. Furthermore, y varies exponentially if and only if $\mathrm{Log}[y]$ *varies linearly,*

$$y = Y_0\ e^{k\,x} \quad \Leftrightarrow \quad \mathrm{Log}[y] = k\,x + \mathrm{Log}[Y_0]$$

Exercise 5.4 and the **PercentGth** program (of Chapter 5) show how to find an exponential given a fixed percentage change for a fixed change in x. The differential equation $\frac{1}{y}\frac{dy}{dx} = k$ simply says y has the fixed instantaneous percentage change $k \times 100\%$.

A differential equation tells us how the quantity changes instantaneously. If we also know an initial value of the quantity, it is intuitively clear that this "start plus change" determines where you go, though it may *not* be entirely clear *how* it determines it.

Mathematically, a continuous dynamical system is the "operation" of going from the "where you start and how you change" to a function of t. We will see that the constant percentage change system has the solution

$$\begin{aligned} y[0] &= Y_0 \\ dy &= k\,y\;dt \\ &\to \qquad y[t] = Y_0\,e^{k\,t}, \quad t \in [0, \infty) \end{aligned}$$

Without knowing this, we can approximate this "operation" by solving a discrete system that moves in small steps of size δt

$$\begin{aligned} y[0] &= Y_0 \\ y[t + \delta t] &\approx y[t] + k\,y[t]\;\delta t \\ &\to \qquad \{y[0], y[\delta t], y[2\delta t], y[3\delta t], \cdots\} \end{aligned}$$

The solution in this case is $y[t] = Y_0(1 + \delta t)^{t/\delta t}$ and gives the very fundamental approximation

$$(1 + \delta t)^{t/\delta t} \approx e^t$$

The general idea is called "Euler's Method" of approximating solutions of initial value problems. We have already seen this idea in several places, beginning with the **SecondSIR** NoteBook in Chapter 2. (Euler's Method for general systems is studied in Chapter 21.) The point of this section is to see that the differential equation gives us a way to work with the function without prior formulas.

You already have an idea of what $y = e^t$ means, so it may seem a little silly to introduce a "definition" for it at this late stage of the game. There may be some gaps in what you know, and we want a definite place to fall back to when the problems get more difficult. For example, later we will want to compute $e^{i\,t}$, where $i = \sqrt{-1}$. Many important functions in higher mathematics are characterized by their differential equation, so this is the first time you will see something that is quite powerful.

The official theory is only important when we get to a question we cannot answer with facts from high school and simple differentiation formulas. What do we mean by e^π or even $3^{\sqrt{2}}$? Certainly, e^3 means "multiply e times itself 3 times," but you cannot multiply e times itself π times - that makes no sense. You probably do not want to believe that - because you can use your calculator for an approximate answer.

When we use calculators to approximate e^π we raise the approximate base $e \approx 2.71828$ to the approximate power $\pi \approx 3.14159$. This implicitly assumes that the y^x-button on our calculator is continuous in both inputs. In other words, the small errors in both e and π only produce a small error in the approximate output to e^π. Does your calculator produce six significant digits of e^π when you put 6 digits of accuracy in for e and π? This is a tough question because you have to decide what is exact. Similarly, an approach to exponentials based on

$$\lim_{x\to\pi,\ y\to e} y^x = e^\pi$$

as both $y \to e$ and $x \to \pi$ is a very difficult way to build a basic theory. It is "natural" in some ways but technically too hard. (You will use differentials to prove it later.)

Definition 8.1 *The Official Natural Exponential Function*
The function

$$y = \mathrm{Exp}[t]$$

is officially defined to be the unique solution of the initial value problem

$$\begin{aligned} y[0] &= 1 \\ dy &= y\ dt \end{aligned}$$

If we use the (unproved) formula for the derivative, we can see that the natural exponential function $y[t] = e^t$ satisfies this differential equation and initial condition because $\frac{dy}{dt} = \frac{de^t}{dt} = e^t = y$ and $e^0 = 1$.

The general Euler's Method is a simple idea once you know the increment approximation from the Definition 5.3 of the derivative. When our function is $f[x]$, we write this approximation

$$f[x + \delta x] = f[x] + f'[x]\ \delta x + \varepsilon \cdot \delta x$$

where the error $\varepsilon \approx 0$ is small when $\delta x \approx 0$ is small. Our function now is $y = y[t]$, so the approximation becomes

$$y[t + \delta t] = y[t] + y'[t]\ \delta t + \varepsilon \cdot \delta t$$

where $\varepsilon \approx 0$ is small when $\delta t \approx 0$ is small. We do not have a formula for $y[t]$, but we do have the value of $y[0] = 1$ and a formula for $y'[t] = y[t]$ given in terms of y. This gives us

Approximate Solution of $y[0] = 1$ & $\frac{dy}{dt} = y$:

$$\begin{aligned} y[t + \delta t] &= y[t] + y[t]\ \delta t + \varepsilon \cdot \delta t \\ y[t + \delta t] &\approx y[t] + y[t]\ \delta t = y[t](1 + \delta t) \qquad \text{when } \delta t \approx 0 \end{aligned}$$

so

$$\begin{aligned} y[\delta t] &\approx y[0] \cdot (1 + \delta t) = (1 + \delta t) \\ y[\delta t + \delta t] &\approx y[\delta t] \cdot (1 + \delta t) = (1 + \delta t)^2 \\ y[2\delta t + \delta t] &\approx y[2\delta t] \cdot (1 + \delta t) = (1 + \delta t)^2 \cdot (1 + \delta t) = (1 + \delta t)^3 \end{aligned}$$

In general, we see that if $y[0] = 1$ and $\frac{dy}{dt} = y$, then

$$y[t] \approx (1 + \delta t)^{(t/\delta t)} \qquad \text{for } t = 0, \delta t, 2\delta t, 3\delta t, \cdots$$

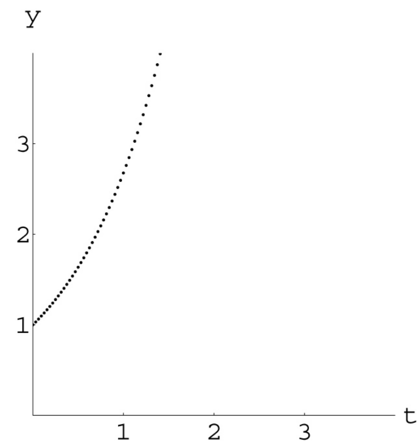


Figure 8.1:2: Euler's approximation $e^t \approx (1+\delta t)^{(1/\delta t)}$

A very basic fact of mathematics says

$$\lim_{\delta t \to 0} (1+\delta t)^{1/\delta t} = e$$

This is a special case of the convergence of the solution of our discrete dynamical system to the solution of the continuous one because $y[1] = e^1 = e$.

We can summarize the section with the formula

$$Y_0 (1 + k\,\delta t)^{t/\delta t} \quad \approx \quad Y_0\ e^{k\,t}$$

The formula on the left can be computed by hand for $t = 0, \delta t, 2\delta t, \cdots$, if necessary, and it comes straight from the initial value problem.

Exercise Set 8.1

1. *Compare the computer's built in function* Exp$[t]$ *to the Euler approximation of the official definition,* $(1+\delta t)^{t/\delta t}$*, for* $\delta t = 1/2, 1/4, 1/16, 1/256$*. Graph both and compare them numerically. How large is the difference between* Exp$[1]$ *and the approximate* $y[1]$ *when* $\delta t = 1/256$*?*

   Now we want you to use the idea that gave us the approximation $(1+\delta t)^{t/\delta t} \approx e^t$ *to approximate the solution of a more general exponential law.*

2. *Approximate Solution of* $\dfrac{dy}{dt} = k\,y$ *with* $y[0] = Y_0$*:*

(a) *Show that* $y[t] = Y_0\,(1 + k\,\delta t)^{t/\delta t}$ *(for* $t = 0, \delta t, 2\delta t, 3\delta t, \cdots$ *) is an approximate solution to the initial condition and differential equation*

$$y[0] = Y_0$$
$$\frac{dy}{dt} = k\,y$$

(b) *Test your approximation numerically and graphically for the special case*

$$y[0] = 3$$
$$\frac{dy}{dt} = -2\,y$$

which has the exact solution $y[t] = 3\,Exp[-2\,t]$.

Here is some help with the exercise. First and foremost, recall the microscope approximation of Definition 5.2 and apply it to the (unknown) function $y[t]$. *Discarding the error term yields an approximation:*

$$y[t + \delta t] = y[t] + ?? + \varepsilon \cdot \delta t \approx y[t] + ???$$

Next, use the fact that $y'[t] = k\,y[t]$ *and substitute this into the microscope approximation,*

$$y[t + \delta t] = y[t] \cdot (??)$$

We know $y[0] = Y_0$*, the initial condition. To find* $y[\delta t]$*, use your approximation*

$$y[\delta t] \approx Y_0 \cdot ??$$
$$y[2\delta t] \approx y[\delta t + \delta t] = y[\delta t] \cdot ??$$
$$y[3\delta t] \approx y[2\delta t + \delta t] = y[2\delta t] \cdot ??$$

Simplification yields the desired result.

3. *Log Linearity*
Show that if y *grows at a constant percentage rate with respect to* x*, then the quantity* $z = Log[y]$ *is a linear function of* x*,* $z = k\,x + Z_0$*. Give the value of* Z_0 *in terms of* $Y_0 = y[0]$.

8.2 e as a "Natural" Base

The number $a = e$ *makes* $y = a^x$ *satisfy* $\frac{dy}{dx} = y$*. Similarly,* $y = e^{k\,x}$ *satisfies* $\frac{dy}{dx} = k\,y$*, and* $y = b^x$ *satisfies* $\frac{dy}{dx} = k\,y$ *provided* $b = e^k$.

All exponential bases are not created equal. All exponential functions $y = b^t$ satisfy

$$y[0] = 1$$
$$\frac{dy}{dt} \propto y$$

but the base with constant of proportionality 1 is $b = e$. This makes e the "natural" base from the point of view of calculus.

Exercise Set 8.2

1. *Let $y = b^t$ for an unknown (but fixed) positive constant b. Use the Chain Rule (see Section 6.4) to show that $y[t]$ satisfies*

$$y[0] = 1$$
$$\frac{dy}{dt} = k\,y$$

What is the value of the constant k?

2. *Show that $y = e^{k\,t}$ satisfies*

$$y[0] = 1$$
$$\frac{dy}{dt} = k\,y$$

If the constant k is the same as in the first part, how much is e^k in terms of b?

3. *Solve the initial value problem*

$$y[0] = 5$$
$$\frac{dy}{dt} = y$$

4. *Solve the initial value problem*

$$y[0] = 5$$
$$\frac{dy}{dt} = k\,y$$

where $k = \mathrm{Log}[2]$. Show that your solution may also be written as $y = 5 \cdot 2^t$

(See the program **ExpEquns**.)

The moral of this exercise is this: We *could* write solutions of initial value problems

$$y[0] = Y_0$$
$$\frac{dy}{dt} = k\,y$$

as $y = Y_0 \cdot b^t$, where $b = e^k$ for $\mathrm{Log}[b] = k$; but, for the purposes of calculus, it is "more natural" to write them in the form $y = Y_0 e^{k\,t}$,

$$y[0] = Y_0$$
$$dy = k\,y\;dt$$
$$\rightarrow \qquad y[t] = Y_0 \cdot e^{k\,t}, \quad t \in [0, \infty)$$

We want you to put this to work in the next problem.

Problem 8.1 *The Canary's Postmortem*

Let $T = T_0 e^{-kt}$ for unknown positive constants T_0 and k. Show that

$$\frac{dT}{dt} = -kT$$

by using the Chain Rule, $T = T_0 e^u$ with $u = -kt$, so $\frac{dT}{du} = T_0 e^u$ and $\frac{du}{dt} = -k$. Express $\frac{dT}{dt}$ in terms of the dependent variable T.

*The value of $e^0 = 1$, so $T = T_0$ when $t = 0$. Show that the function T solves the cooling problem of Exercise 4.2.1 for certain choices of the constants T_0 and k. How much is T_0 in that problem? How could you find the constant k so $T = 60$ when $t = 10$? (*HINT: *Solve $60 = 75\, e^{-10\,k}$ using log.)*

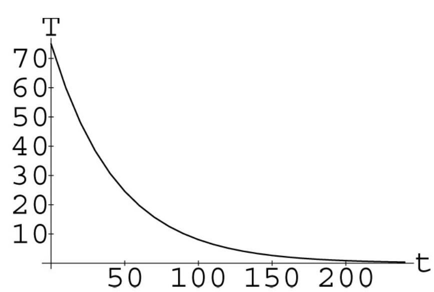


Figure 8.2:3: The cooling canary

*Graph your specific function $T = T_0\, e^{-k\,t}$ using the computer and verify that the temperature at time 10 is 60. (See the program **ExpEquns**.)*

In the saga of the frozen canary, we let the outside temperature be zero. This simplifies the math (and, of course, freezes the poor dead canary). The next problem has you generalize the law of cooling to an arbitrary ambient temperature. We want you to explain why the initial value problem:

$$T[0] = T_0$$
$$\frac{dT}{dt} = k\,(T_a - T)$$

is a reasonable model of temperature adjustment toward ambient - either warming or cooling. We also want you to find an analytical solution to the model.

Problem 8.2 *Newton's Law of Cooling for Ambient Temperature T_a*

Suppose a small object is placed in a large room with constant room temperature T_a - the constant ambient temperature. The small inanimate object (that does not heat itself or evaporate, etc.) is placed in the room at a different initial temperature, T_0, and cools or warms toward the room temperature. This problem helps you formulate Newton's law of cooling for non-zero ambient temperature.

1. *Let T be the (variable) temperature of your object and t be the time. Choose units of your liking. The derivative $\frac{dT}{dt}$ represents the instantaneous rate of increase in the temperature. Why?*
2. *How do we represent warming and cooling in terms of $\frac{dT}{dt}$?*
3. *Suppose we put a covered cup of almost boiling water in a normal room, so $T_a = 21°C$ and $T_0 = 100°C$. Initially, the object cools at a fast rate ($\frac{dT}{dt}$ is a large magnitude negative number), while as T approaches $T_a = 21$, the rate of cooling slows.*
   (a) *What is the value of $(T_a - T)$ when $T = 100$? How much "cooling" is happening at this instant if $\frac{dT}{dt} = k\,(T_a - T)$?*
   (b) *How much "cooling" is happening at the instant when $T = 22$ if $\frac{dT}{dt} = k\,(T_a - T)$?*
4. *Suppose we put a covered cup of almost freezing water in a normal room, so $T_a = 21°C$ and $T_0 = 0°C$. Initially, the object warms at a fast rate, while as T approaches $T_a = 21$, the rate of warming slows.*
   (a) *What is the value of $(T_a - T)$ when $T = 0$? How much "warming" is happening at this instant if $\frac{dT}{dt} = k\,(T_a - T)$?*
   (b) *How much "warming" is happening at the instant when $T = 19$ if $\frac{dT}{dt} = k\,(T_a - T)$?*

5. *Use the previous parts of this problem to explain how the initial value problem*

$$T[0] = T_0$$
$$\frac{dT}{dt} = k\,(T_a - T)$$

describes temperature change. Write something like, "The thing that causes the temperature to change is amount away from ambient temperature of the object ... and the temperature adjusts toward ambient. ... This is because ...."

Now that you understand why the initial value problem describes temperature change, solve the mathematical problem analytically. One approach is to change variables to make the ambient law of cooling mathemaitcally the same as the canary law with zero ambient temperature. The physical meaning of U in the next problem is 'the normalized temperature' or temperature away from ambient.

Problem 8.3 *One Solution of* $T[0] = T_0 \quad \& \quad \frac{dT}{dt} = k\,(T_a - T)$

Let T_a, T_0 and k be constants. Let T and U be dependent variables of the independent variable t.

1. Let $U = T - T_a$ or $T = U + T_a$ and show that the following are equivalent initial value problems:

$$T[0] = T_0 \qquad \Leftrightarrow \quad U[0] = T_0 - T_a$$
$$\frac{dT}{dt} = k(T_a - T) \qquad \frac{dU}{dt} = -k\,U$$

2. Show that the solution of

$$U[0] = U_0$$
$$\frac{dU}{dt} = -k\,U$$
$$\text{is } U[t] = U_0\,e^{-k\,t}$$

3. Let $T[t] = U[t] + T_a$, where $U[t]$ is the solution to the previous part of this problem and $T_0 = U_0 + T_a$. Show that $T[t]$ satisfies

$$T[0] = T_0$$
$$\frac{dT}{dt} = k\,(T_a - T)$$
$$\text{and may be written } T[t] = T_a + (T_0 - T_a)\,e^{-k\,t}$$

4. Prove that $T[0] = T_0$ and $\lim_{t\to\infty} T[t] = T_a$. How could you see the limit directly from the differential equation?

Another approach to the analytical solution is to just "plug in."

Problem 8.4 *The 'Method' of Unknown Constants*

Let a, b, and c be constants.

1. (a) Substitute the function $y = a + b\,e^{c\,x}$ into the differential equation

$$\frac{dy}{dx} = c\,(y - a)$$

and show that it is a solution for any value of the constant b.

(b) Substitute the function $y = a + b\,e^{c\,x}$ into the condition

$$y[0] = 5$$

and show that you must have $b = 5 - a$.

(c) Substitute the function $y = a + b\,e^{c\,x}$ into the initial value problem

$$y[0] = Y_0$$
$$\frac{dy}{dx} = c\,(y - a)$$

and express b in terms of the constants a, c and Y_0.

The story of the fallen tourist in subsection 4.2.2 led to the differential equation

$$\frac{dD}{dt} = k\,D$$

for some positive constant k, where D is the distance an object has fallen and t is time. This differential equation says, "The farther you fall, the faster you go." in a specific way. In a general sense this is true, but it cannot be specifically a linear function of D. We called this Galileo's first conjecture because it gives rise to the Bugs Bunny Law of Gravity: If you don't look down, you don't fall. We want you to show why.

Problem 8.5 *Bugs Bunny's Law of Gravity*

*How does Bugs' Law say, "The farther you go, the faster you fall?" Prove that an object released from $D = 0$ at $t = 0$ does not fall under the Bugs Bunny Law of Gravity: $\frac{dD}{dt} = k\,D$, $k > 0$ constant. (See the program **ExpEquns**.)*

In Problem 21.8, we will look at Wiley Coyote's Law of Gravity: $\frac{dD}{dt} = k\sqrt{D}$, $k > 0$ constant. This law gives a correct prediction to the position of an object falling under gravity, but has the strange property that, "you don't fall *until* you look down." (It is not a well-posed physical law because of mathematical non-uniqueness.)

Galileo's simple law $\frac{d^2D}{dt^2} = g$, g a constant, is studied in Exercise 10.2. It is actually simpler than either Bugs' Law or Wiley's. And it's correct, too, in vacuum. The Bungee Diver Project extends this to falling in air by using Newton's extension of Galileo's law.

8.3 Growth of Log, Exp, and Powers

When two functions both tend to infinity as x tends to infinity, one may grow "faster." This section shows how to measure of their "order of infinity" as the limit of their ratio.

Our main aim in this section is to see that logs tend to infinity "much slower" than powers, whereas exponentials tend to infinity "much faster" than powers. First, let us think a little about the "eventual growth" of some familiar high school functions.

Example 8.1 $y = x^2$ *vs.* $y = x^2 + 200x + 3000$

At a modest scale, the function $f[x] = x^2 + 200x + 3000$ is bigger than $g[x] = x^2$. For example, see Figure 8.3:4.

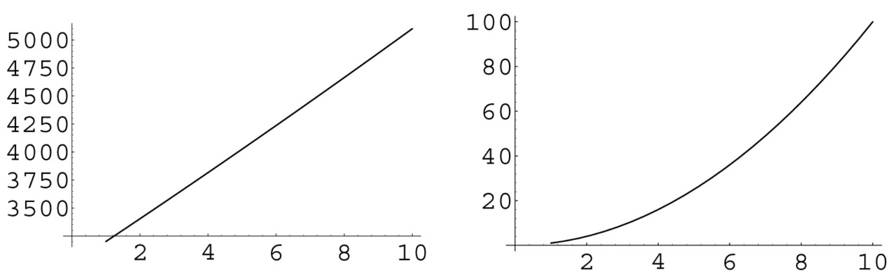
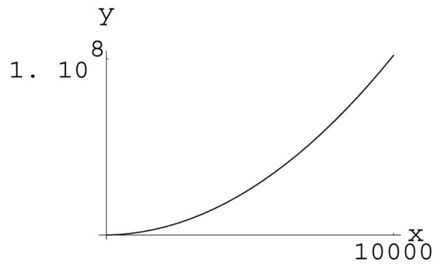


Figure 8.3:4: $y = x^2 + 200x + 3000$ vs. $y = x^2$ on $1 < x < 10$

At a larger scale, these functions look alike, as in Figure 8.3:5

In terms of the "eventual" size of the two functions, we can compare by taking the limit of the



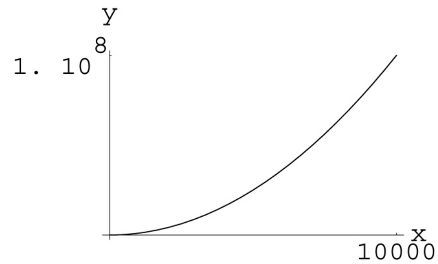


Figure 8.3:5: $y = x^2 + 200x + 3000$ vs. $y = x^2$ on $1 < x < 10000$

ratio

$$\begin{aligned}
\lim_{x\to\infty} \frac{f[x]}{g[x]} &= \lim_{x\to\infty} \frac{x^2 + 200x + 3000}{x^2} \\
&= \lim_{x\to\infty} \frac{x^2/x^2 + 200x/x^2 + 3000/x^2}{x^2/x^2} \\
&= \lim_{x\to\infty} \frac{1 + 200/x + 3000/x^2}{1} \\
&= \lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{200}{x} + \lim_{x\to\infty} \frac{3000}{x^2} \\
&= 1 + 0 + 0
\end{aligned}$$

Figure 8.3:6 shows the graph of the ratio.

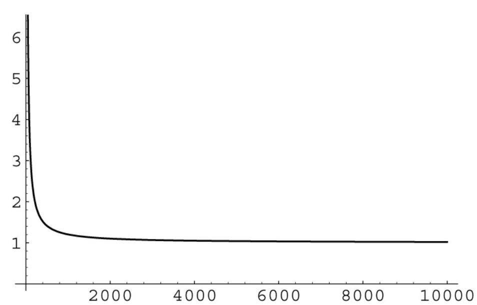


Figure 8.3:6: $y = \left(x^2 + 200x + 3000\right)/x^2$ on $1 < x < 10000$

Definition 8.2 *Order of Infinity*

When two functions tend to infinity, $f[x] \to \infty$ and $g[x] \to \infty$, but the ratio tends to a non-zero amount, $\lim_{x\to\infty} \frac{f[x]}{g[x]} = a$ for $0 < a < \infty$, we say both functions grow at the "same order of infinity."

In general if $f[x] = a\ x^p +$ "lower power terms" the growth at infinity is the same as $g[x] = x^p$. Precisely, the limit of the ratio $f[x]/g[x]$ is just the number a. Different powers, however, grow at different rates.

Example 8.2 $y = 500x$ *vs.* $y = x^2/500$

The linear function $f[x] = 500x$ is much larger than $g[x] = x^2/500$ at small scale,

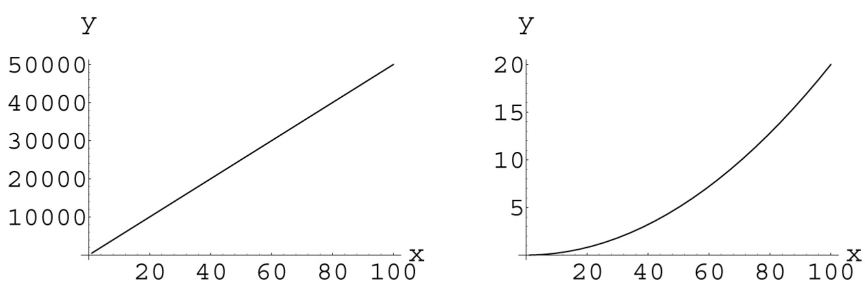


Figure 8.3:7: $y = 500x$ vs. $y = x^2/500$ on $0 < x < 100$

Figure 8.3:8 shows that, at a large scale, the quadratic function grows faster.

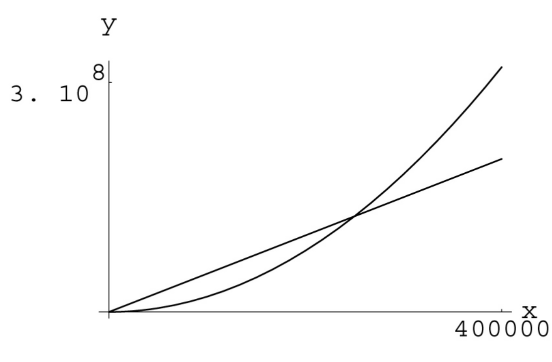


Figure 8.3:8: $y = 500x$ vs. $y = x^2/500$ on $0 < x < 400000$

The "eventual" size of the two functions can be compared by taking the limit of the ratio

$$\begin{aligned}\lim_{x\to\infty}\frac{f[x]}{g[x]} &= \lim_{x\to\infty}\frac{500x}{x^2/500}\\ &= \lim_{x\to\infty}\frac{250000x}{x^2}\\ &= 250000\lim_{x\to\infty}\frac{1}{x}\\ &= 0\end{aligned}$$

In this case, the higher power is "winning" because the denominator is big, thereby forcing the ratio to be small. Figure 8.3:9 shows the graph of the ratio.

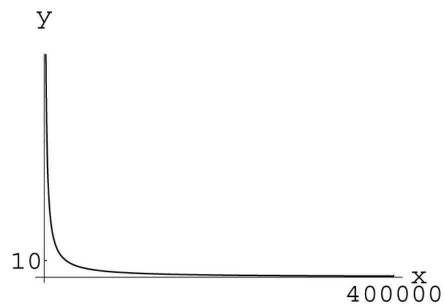


Figure 8.3:9: $y = \frac{500x}{x^2/500}$ on $0 < x < 400000$

The ratio the other way around gives

$$\begin{aligned}\lim_{x\to\infty}\frac{g[x]}{f[x]} &= \lim_{x\to\infty}\frac{x^2/500}{500x}\\ &= \lim_{x\to\infty}\frac{x^2}{250000x}\\ &= \lim_{x\to\infty}\frac{x}{250000}\\ &= \infty\end{aligned}$$

In this example, the function $g[x] = x^2/500$ has a higher order of infinity than $f[x] = 500x$ because "eventually" it is larger in the strong sense that the ratio even tends to infinity. It is eventually twice as big, three times as big, and so forth.

Exercise 8.3.2 shows that higher powers have higher orders of infinity. In particular, roots, $x^{1/n}$ have a lower order of infinity than integer powers x^n, as in Figure 8.3:10.

Logs have an order of infinity below *all* roots, while exponentials have an order above every power. Specifically, two important facts about the rate of growth of the natural log and exponential functions are as follows:

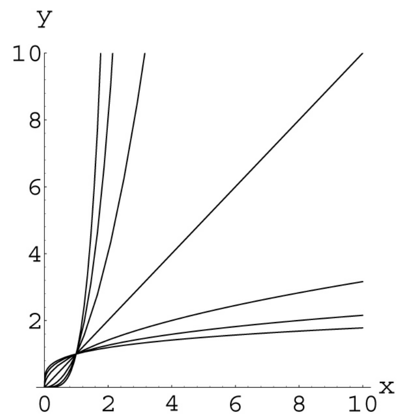


Figure 8.3:10: $y = x^n$ and $y = x^{1/n}$, $n = 1, 2, 3, 4$

Theorem 8.3 *Orders of Infinity*

Let p be any positive real number. Then

$$\lim_{t\to\infty} \frac{e^t}{t^p} = \infty \qquad \textit{and} \qquad \lim_{t\to\infty} \frac{t^p}{\mathrm{Log}[t]} = \infty$$

or, equivalently,

$$\lim_{t\to\infty} \frac{t^p}{e^t} = 0 \qquad \textit{and} \qquad \lim_{t\to\infty} \frac{\mathrm{Log}[t]}{t^p} = 0$$

The exponential beats any power, and any power beats the logarithm to infinity.

Proof Based on High School Math:

First we will consider the ratio $\frac{t^p}{e^t}$ for integer values of t, $\frac{n^p}{e^n}$, for $n = 1, 2, \cdots$. It is sufficient to show that the p^{th} root of the ratio tends to zero,

$$\frac{n}{e^{n/p}} \to 0$$

by continuity of positive powers at zero. Let $b = e^{1/p}$, and notice that $b > 1$ and so $b^{1/2} > 1$ as well. Write

$$b^{1/2} = 1 + a$$

for $a > 0$.

The binomial theorem says

$$b^{n/2} = (1 + a)^n = 1 + na + \cdots + a^n > na$$

since $a > 0$, so that $e^{n/p} = b^n > n^2 a^2$. Hence, we have

$$\frac{n}{e^{n/p}} < \frac{n}{n^2 a^2} = \frac{1}{n a^2} \to 0 \qquad \text{as} \quad n \to \infty$$

For continuous values of $t > 0$, there is always an integer satisfying $n - 1 < t \leq n$. For these values, we have

$$\frac{t^p}{e^t} < \frac{n^p}{e^{n-1}} = e\,\frac{n^p}{e^n} \to 0$$

Example 8.3 $2^\infty/\infty^3$

There are many other "orders of infinity" or "orders of infinitesimal" that we can deduce from the basic result given in the preceding theorem. For example, what is

$$\lim_{x\to\infty} \frac{2^x}{x^3} =?$$

First, we can write $2 = e^k$, where $k = \text{Log}[2]$. (Simply take logs of both sides of $2 = e^k$.) Our limit becomes

$$\lim_{x\to\infty} \frac{e^{k\,x}}{x^3} =?$$

for a positive k. ($\text{Log}[2] = k \approx 0.693147806$.)

We consider the k^{th} root of our limit to obtain

$$\begin{aligned}
\lim_{x\to\infty} \sqrt[k]{\frac{e^{k\,x}}{x^3}} &= \lim_{x\to\infty} \left(\frac{e^{k\,x}}{x^3}\right)^{1/k} \\
&= \lim_{x\to\infty} \frac{e^{k\,x/k}}{x^{3/k}} \\
&= \lim_{x\to\infty} \frac{e^x}{x^p} = \infty
\end{aligned}$$

with $p = 3/k$, by the theorem. Since positive roots only tend to infinity when the quantity tends to infinity, we have

$$\lim_{x\to\infty} \frac{2^x}{x^3} = \infty$$

Example 8.4 $2^{-\infty}/\infty^3$ *or* $\lim_{x\to-\infty} \frac{2^x}{x^3} = 0$

Consider the limit

$$\lim_{x\to-\infty} \frac{2^x}{x^3} =?$$

First, change to natural base.

$$\lim_{x\to-\infty} \frac{e^{k\,x}}{x^3} = ?$$

with $k = \mathrm{Log}[2] > 0$. Next, replace x by $u = -x$,

$$\lim_{u\to+\infty} \frac{e^{-k\,u}}{(-u)^3} = -\lim_{u\to+\infty} \frac{e^{-k\,u}}{u^3}$$

Now, notice that the limit is obvious by some arithmetic,

$$-\lim_{u\to+\infty} \frac{e^{-k\,u}}{u^3} = -\lim_{u\to+\infty} \frac{1}{u^3\,e^{k\,u}} = 0$$

because both terms in the denominator tend to infinity. (We can also see this limit by "plugging in" the limiting values.)

Example 8.5 $-\infty^3 \cdot 2^{-\infty}$

The limit

$$\lim_{x\to-\infty} x^3\,2^x = ?$$

has one term growing and the other shrinking ($2^{\text{large negative number}}$). Replace $x = -u$ and $2 = e^k$, for $k = \mathrm{Log}[2] > 0$, so we have

$$\begin{aligned}
\lim_{u\to+\infty} (-u)^3\,2^{-u} &= -\lim_{u\to+\infty} u^3\,2^{-u} \\
&= -\lim_{u\to+\infty} \frac{u^3}{2^u} \\
&= -\lim_{u\to+\infty} \frac{u^3}{(e^k)^u} = -\lim_{u\to+\infty} \frac{u^3}{e^{k\,u}} \\
&= 0
\end{aligned}$$

since exponentials beat powers to infinity.

Similar tricks reduce various limits to the logarithmic comparison of the Orders of Infinity Theorem.

Example 8.6 $\lim_{x\downarrow 0} x^x = 1$

We may write $x = e^{\mathrm{Log}[x]}$, so $x^x = e^{x\ \mathrm{Log}[x]}$ and

$$\lim_{x\downarrow 0} x^x = \lim_{x\downarrow 0} e^{x\ \mathrm{Log}[x]} = e^{\lim_{x\downarrow 0} x\ \mathrm{Log}[x]}$$

if the last limit exists. This is a fundamental limit which we compute next.

Example 8.7 $\lim_{x\downarrow 0}\ x\ \mathrm{Log}[x] = 0$

Replace $x = 1/u$ in the limit, so $u \to +\infty$ as $x \downarrow 0$, and the limit

$$\lim_{x\downarrow 0}\ x\ \mathrm{Log}[x]$$

becomes

$$\begin{aligned}\lim_{u\to\infty} \frac{1}{u}\,\mathrm{Log}[\frac{1}{u}] &= \lim_{u\to\infty} \frac{1}{u}\,\mathrm{Log}[u^{-1}] \\ &= -\lim_{u\to\infty} \frac{\mathrm{Log}[u]}{u} \\ &= 0\end{aligned}$$

since powers beat logs to infinity.

Now, apply this result to the x^x limit,

$$\lim_{x\downarrow 0} x^x = e^{\lim_{x\downarrow 0} x\ \mathrm{Log}[x]} = e^0 = 1$$

This limit is a reason to write $0^0 = 1$.

Exercise Set 8.3

Use the program ***Infinities*** *to explore orders of infinity and compute limits by computer.*

1. *Ratios of Powers*
   Show that each of the functions $g[x]$ tends to infinity faster than the corresponding $f[x]$:

   a) $g[x] = x^5$ and $f[x] = x^3$ *b)* $g[x] = x^2$ and $f[x] = \sqrt{x}$

   c) $g[x] = \sqrt{x}$ and $f[x] = \sqrt[3]{x}$ *d)* $g[x] = \frac{x^5}{10}$ and $f[x] = 10x^3$

   e) $g[x] = \frac{x^2}{2}$ and $f[x] = 2\sqrt{x}$ *f)* $g[x] = \frac{\sqrt{x}}{2}$ and $f[x] = 3\sqrt[3]{x}$

2. *Order of Powers*
   Let $f[x] = a_1 x^{p_1} + b_1 x^{q_1} + c_1 x^{r_1}$ and $g[x] = a_2 x^{p_2} + b_2 x^{q_2} + c_2 x^{r_2}$ for positive constants a_i, b_i, p_i, q_i, r_i. Show that $f[x]$ has a higher order of infinity than $g[x]$ when the highest power of $f[x]$ is greater than the highest power of $g[x]$. What is the limit of the ratio $f[x]/g[x]$? What is the limit if the highest powers agree?

3. *Drill Limits*
   Compute the limits $(p, k > 0)$

$a)$	$\lim_{x\to 0} x^p \,\mathrm{Log}[x]$	$b)$	$\lim_{x\to\infty} x^{1/x}$
$c)$	$\lim_{\Delta x\downarrow 0} \Delta x^{1/\,\mathrm{Log}[\Delta x]}$	$d)$	$\lim_{x\to\infty} x^p \,\mathrm{Log}[x]$
$e)$	$\lim_{\delta\to 0} \frac{1+k\,\delta}{\delta}$	$f)$	$\lim_{\delta\downarrow 0} \delta^{\frac{\delta+k}{Log[\delta]}}$

4. *Suppose* $b > a > 0$*. Find* $\lim_{x\to\infty} \frac{b^x}{a^x}$

5. *Negative? Exponentials*
   If the base of an exponential function $f[x] = b^x$ *satisfies* $0 < b < 1$*, we think of this as a "negative" exponential in the following sense:*

   (a) *Solve for a constant* k *so that* $b^x = e^{k\,x}$ *for all* x*.*
   (b) *Show that* k *from part (a) is negative when* $0 < b < 1$*.*
   (c) *Show that* $\lim_{x\to\infty} b^x = 0$ *when* $0 < b < 1$*.*

6. *Use logs to prove the other half of the Orders of Infinity theorem above. If we wish to show that* $\lim_{x\to\infty} \dfrac{x^p}{\mathrm{Log}[x]} = \infty$*, then change variables by letting* $u = \mathrm{Log}[x]$*, so* $x = e^u$*. We show show instead that*
   $$\lim_{u\to\infty} \frac{(e^u)^p}{u} = \infty$$
   by taking p^{th} *roots. Why does this establish the result?*

7. *All Exponentials Beat All Powers*
   Suppose $b > 1$ *and* $p > 0$*. What are the values of the limits*
   $$\lim_{x\to\infty} \frac{b^x}{x^p} = ? \qquad \text{and} \qquad \lim_{x\to\infty} \frac{x^p}{b^x} = ?$$

8.4 Official Properties

The important functional identity $e^x\ e^y = e^{x+y}$ *follows from the differential equation defining the exponential function.*

In the Mathematical Background at www.math.uiowa.edu/~stroyan/InfsmlCalculus/FoundationsTOC.htm, we prove a general existence and uniqueness result for continuous dynamical systems. The proof includes showing that Euler's approximation converges as the step size Δt tends to zero. Mathematical uniqueness of the solution to an initial value problem is what makes dynamical systems deterministic scientific models. Uniqueness is really what you are thinking of when you say, "***The*** solution of this system models $\cdots$." Uniqueness is also mathematically important. For now, we will use the following result:

Theorem 8.4 *Unique Solution to a Linear Dynamical System*

For any real constants Y_0 and k, there is a unique real function $y[t]$ defined for all real t satisfying

$$y[0] = Y_0$$
$$\frac{dy}{dt} = k\,y$$

This function can be written $y[t] = Y_0\ e^{k\,t}$.

Simply put, the theorem means that saying $y[t] = Y_0\ e^{k\,t}$ is *exactly* the same thing as saying $y[0] = Y_0$ and $\frac{dy}{dt} = k\,y$. The theorem assures us in particular that a function $\mathrm{Exp}[t]$ of our official definition exists. We know intuitively from our experience with programs like **SecondSIR** and **EulerApprox** that the computer approximations do converge. The next subsection shows how the important addition formula for exponentials follows from uniqueness of the solution to our official definition.

8.4.1 Proof that $e^c\,e^t = e^{(c+t)}$

Let $C = \mathrm{Exp}[c]$, and consider the function $y[t] = C\ \mathrm{Exp}[t]$. We know $y[0] = C$ and by the Superposition Rule for differentiation that $\frac{dy}{dt} = C\ \mathrm{Exp}[t] = y$. This means that y is a solution to

$$y[0] = C$$
$$\frac{dy}{dt} = y$$

Now consider the function $z[t] = \mathrm{Exp}[c+t]$. Again, $z[0] = \mathrm{Exp}[c] = C$ and the Chain Rule says $\frac{dz}{dt} = \mathrm{Exp}[c+t] = z$, so z is also a solution to

$$z[0] = C$$
$$\frac{dz}{dt} = z$$

This is the same dynamical system, written with a different letter. Uniqueness means that both $y[t]$ and $z[t]$ are the same function

$$\mathrm{Exp}[c]\ \mathrm{Exp}[t] = \mathrm{Exp}[c+t]$$

and proves the functional identity of the natural exponential function.

Our new definition causes us a technical problem. We know how to compute rational powers of e. We want to show that the new definition extends what we already know.

Problem 8.6 *Let e be the number $Exp[1]$, that is, the value of the solution of the dynamical system at time 1.*

1. *Show that $e^2 = Exp[2]$.*
2. *Show that $e^{1/3} = Exp[1/3]$ by letting $a = Exp[1/3]$ and computing $a \times a \times a$.*
3. *Show that $e^q = Exp[q]$ for any rational $q = m/n$.*

8.5 Projects

8.5.1 Numerical Computation of $\frac{da^t}{dt}$

There are two small Projects that will help you understand the natural base. The first has you use the computer to directly compute the derivative of b^t. You will see that this leads to

$$\frac{db^t}{dt} \propto b^t$$

but it does not give an obvious value to the constant of proportionality. However, you can experiment with your computations and find the value of b that makes this constant one. This is $b = e \approx 2.71828\cdots$.

8.5.2 The Canary Resurrected - Cooling Data

The second mathematical project in that chapter asks you to compare some actual cooling data with the prediction of Newton's law of cooling. It is an interesting scientific project for you to measure this yourself and we believe that you can do better than the first student data we present if you wish to try. (We are not sure if the students warmed the cup before they started measurements. You will see how this shows up in their data.)