Chapter 9
Graphs and the Derivative

The computer draws beautiful graphs and does not force us to think very much, so why use calculus to draw graphs?

The translation between formulas and graphs and the interpretation of graphs themselves are important parts of this course. Graphs often reveal mathematical results simply and clearly, but graphs do this with "trends" or "shapes," not just points. Calculus with the computer will give us information that includes both points and trends.

In real applications, the scale of a plot can be far from obvious. Hot objects radiate; you have heard of "red hot." Planck discovered that the intensity of radiation at frequency ω of a black body at absolute temperature T is

$$I = \frac{\hbar\omega^3}{\pi^2 c^2 (e^{\hbar\omega/(kT)} - 1)}$$

where $\hbar \approx 6.6255 \times 10^{-27}$ (erg sec) is Planck's constant, $c \approx 2.9979 \times 10^{10}$ (cm/sec) is the speed of light, and $k \approx 1.3805 \times 10^{-16}$ (erg/deg) is Boltzman's constant. The frequency ω where this function peaks, is the "color" of the radiation we observe at temperature T. This peak predicts the empirically observed law of radiation discovered earlier by Wein.

If we want to find this peak by graphing, we have a big difficulty. What scale should we use? The constants c, k, and $\hbar$ cover 37 orders of magnitude, whereas a single graph can scarcely show more than two. This is a complicated formula with messy constants; but aside from the technical difficulty, calculus can help, even in just finding the interesting scale of the graph. This is taken up in the project on Planck's Formula.

Calculus helps you set the scales, which usually are not so obvious in real applications. Calculus finds formulas for geometric features of interest. Calculus finds the qualitatively interesting range to plot and once this is found, the computer can make a quantitatively accurate picture. Again, our goal is for you to form a nonlinear combination

$$\begin{aligned}&\text{Knowledge[calculus + computing]} > \\ &\qquad\qquad \text{Knowledge[calculus]} + \text{Knowledge[computing]}\end{aligned}$$

The scale of a graph can alter our perception of the behavior of a function. We begin the chapter with a look at graphing without knowledge of shape.

9.1 Graphs from Formulas

Two simple approaches to graphing are plotting points and using computer packages. The exercises at the end of this section show some limitations of these methods.

Plotting points alone is usually a bad way to sketch graphs because that information alone requires many points to construct a shape and a leap of faith that we have connected the points correctly. If we have only numerical data, that is all we can do. Later in this chapter, we will learn to use calculus to tell us shape information, such as where the graph is increasing or decreasing, so that only a few points are required to give qualitatively accurate graphs. This first section is about what goes wrong without this information.

Even with the computer, which will plot 1,000 points if you ask it to, we often can use calculus to decide what range of values contains the important information. Poor choice of scales can come up in innocent or simple-minded ways or as a result of large differences in the size of scientific constants as in the Planck's Formula project.

Example 9.1 *Simple-Minded Scales*

Which of the following is the graph of

$$y = x^5 + 4\,x^4 + x$$

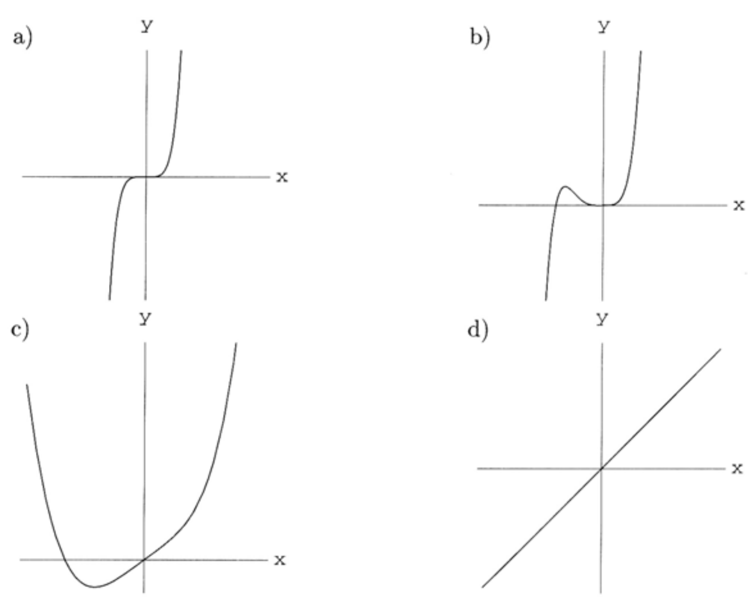


Figure 9.1:1: $y = x^5 + 4x^4 + x$ on four different scales

The answer is all of these are graphs of this same polynomial. Graph (a) is for $-100 < x < 100$. Graph (b) is for $-10 < x < 10$. Graph (c) is for $-1 < x < 1$. Graph (d) is for $-0.1 < x < 0.1$. They appear different because the wiggle in the medium-scale graph is an insignificant part of the term x^5 when $x = 100$. The small unit scale misses some of the medium-size 10 wiggle. Why is the tiny scale straight? (What is the view of the graph in a microscope?) These shape differences are obvious if you compute some sizes but could surprise you if you are using the computer and blindly hoping for "good" graphs. Calculus can find the interesting shape information, and the computer can then draw it accurately. The four graphs above are accurate, but they are drawn on different scales. Try these yourself with the computer.

Exercise Set 9.1

The next exercise shows you a simple reason why points alone are not enough.

1. *Three Points Are Not Enough*

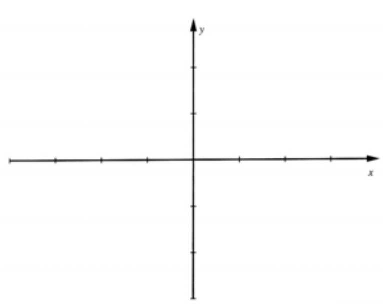
   (a) *The cartesian pairs* $(-2,-2),(0,0),(2,2)$ *are recorded in the table below the blank graph in the next figure. Plot these points.*

   (b) *Show that the graph of* $y = x$ *contains all three points from part (1) and sketch the graph.*

   (c) *Show that the graph of* $y = x^3 - 3x$ *contains all three points from part (1). Can you sketch it without more points?*

   (d) *Find a point on the graph of* $y = x$ *that is not on the graph of* $y = x^3 - 3x$ *and plot it.*

   (e) *Use the graphing program in aComputerIntro to plot both functions on the same graph.*



x		-2		0		2				
y		-2		0		2				

Figure 9.1:2: Three points on two graphs

2. *Find the equations of the lines shown in Figure 9.1:3.*

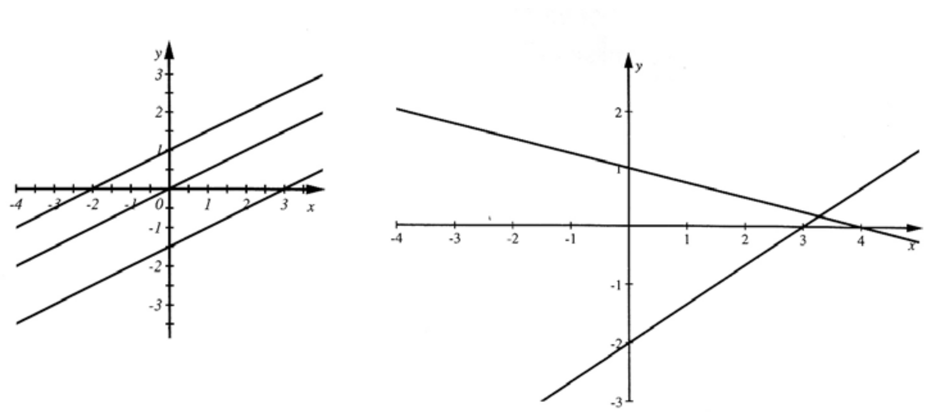


Figure 9.1:3: Three parallel lines and two intersecting lines

3. *Scale of the Plot*

   Use the computer to make several plots on different scales. First, replot

$$y = x^5 + 4\,x^4 + x$$

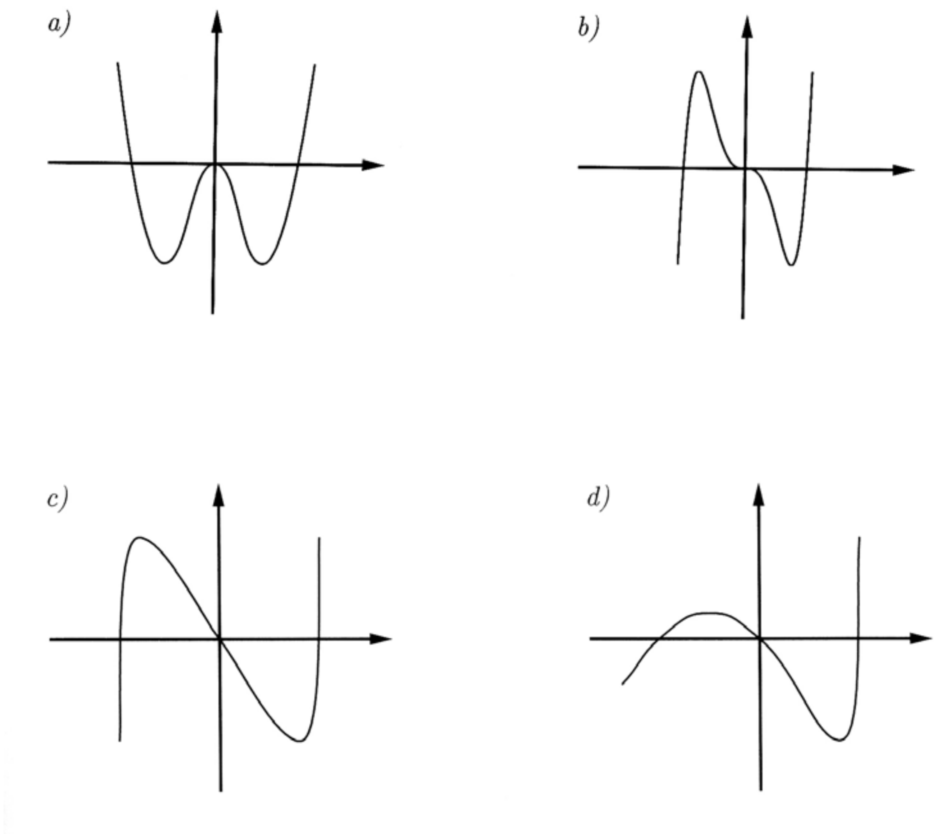
   at the four scales described in Example 9.1:1 above, but leave the default "Ticks" on so that the computer puts the scales on the plots. This will show you more clearly what we described in the previous paragraph. Second, plot the functions below on the different suggested scales:

$$y = x^x 0 < x < 2 \text{ and } 0 < x < 5$$
$$y = 3x^4 - 4x^3 - 36x^2 - 10 < x < 10 \text{ and } -3 < x < 5$$

   Explain why the pairs of graphs appear to be different even though they are of the same function.

In the next exercise, you are given four choices for the graph and asked which one is best. Identify a shape feature related to the derivative (or what you would see in a microscope), and then check the formula to see if it matches. For example, graph (a) is decreasing for large magnitude negative numbers, and graph (b) is increasing. The derivative, $\frac{dy}{dx} = 15(x^2 - 1)x^2$, *is positive if the magnitude of* x *is large; the squares remove the dependence on sign. What does this say about the choice between (a) and (b)? What else can you eliminate?*

4. *Which of the following is the graph of* $y = 3x^5 - 5x^3$*? Which shape features of the incorrect ones makes each one wrong?*



The previous exercise is a little artificial. It really means, "which graph has all the shape information of the algebraic curve." It also implicitly assumes that one of the figures is correct. The point is that calculus tells us the shape or all the "ups and downs" of the curve.

9.2 Graphs Without Formulas

Qualitative information certainly can exist without any formulas.

Graphs are primarily good for qualitative information rather than quantitative accuracy. Graphs readily show where quantities are increasing or decreasing but only give rough approximations to amounts, rates of increase, and so on.

Exercise Set 9.2

Make a qualitative rough sketch of a graph of the distance traveled as a function of time on the following hypothetical trip: You travel a total of 100 miles in 2 hours. Most of the trip is on rural interstate highway at the 65 mph speed limit. (What qualitative feature or shape does the graph of distance vs. time have when speed is 65 mph?) You start from your house at rest, gradually increase your speed to 25 mph, slow down, and stop at a stop sign. (What shape is the graph of distance vs. time while you are stopped?) You speed up again to 25 mph, travel a while and enter the interstate. At the end of the trip, you exit, slow to 25 mph, stop at a stop sign, and proceed to your final destination.

9.3 Ups and Downs of the Derivative

Calculus lets us "look" in a powerful microscope at a graph before we have the whole graph. We must "look" by computing derivatives.

Of course, all we would see in a powerful microscope is the graph of a straight line $dy = m\,dx$ in the microscope (dx, dy)-coordinates (where $m = f'[x]$ with x fixed). This can only be one of the three qualitative shapes shown in Figure 9.3:4.

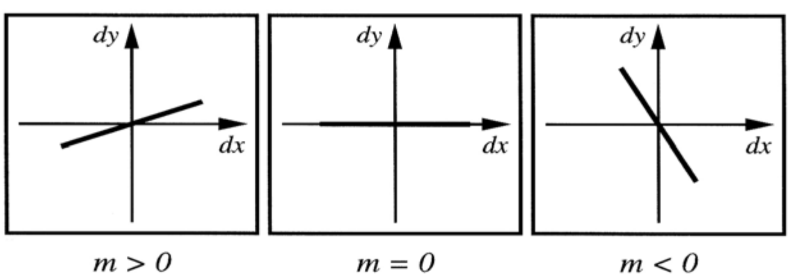


Figure 9.3:4: $dy = m\,dx$ for $m > 0$, $m = 0$, $m < 0$

If $f'[x]$ is undefined, something else may appear in the microscope, and the rules of calculus do not let us "see" in this case. The exact slope can be measured on an auxiliary scale (if necessary), as in Figure 9.3:5.

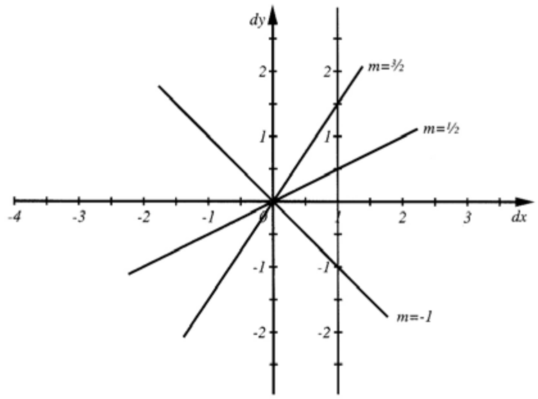


Figure 9.3:5: A Slope Scale

Our approach to graphing will be to fill out a table that looks like the blank graphing table shown in Figure 9.3:6.

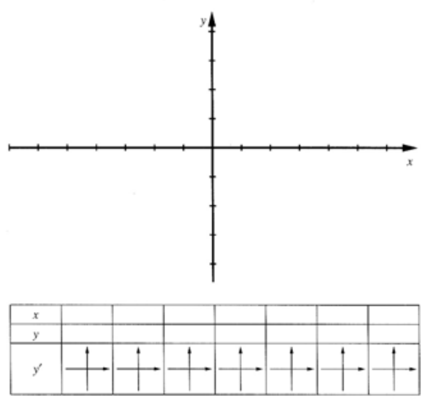


Figure 9.3:6: A blank graphing table

Procedure 9.1 *Graphing $y = f[x]$ with the First Derivative*

1. Compute $f'[x]$ and find **all** values of x where $f'[x] = 0$ (or $f'[x]$ does not exist). Record these in the microscope row as horizontal line segments (or $*'s$ if the derivative does not exist).
2. Check the sign (+) or (-) of $f'[x]$ at values between each of the points from the first part. Record $f'[x] = (+) > 0$ as an upward sloping microscope line and $f'[x] = (-) < 0$ as a downward sloping microscope line.
3. Compute a few key (x, y) pairs (using $f[x]$ to find values of y, not $f'[x]$) and record the numbers on the x and y rows. For example, you should at least compute the (x, y) pairs for the x values used in step 1.
4. Plot the points and mark small tangent lines.
5. Connect the points with a curve matching the tangents as you pass through the points and increasing or decreasing according to the table between the horizontal points.

Do not start by plotting points. Start with the "shape" information of the microscope row so you first find out which points are interesting to plot.

If you are plotting with the computer, you can just work steps (a) and (b). This will tell you the range of x values to plot.

Example 9.2 *Graphing $y = f[x] = 3\,x^5 - 5\,x^3$ from Scratch*

First, compute $f'[x]$,

$$y' = 3 \cdot 5 \cdot x^4 - 5 \cdot 3 \cdot x^2 = 15\,x^4 - 15\,x^2 = 15\,x^2\,(x^2 - 1)$$

This derivative is defined for all real values of x.

Second, find all places, x, where $y' = 0$,

$$0 = 15\,x^2\,(x^2 - 1) \qquad \Leftrightarrow \qquad x = 0 \quad \text{or} \quad x = +1 \quad \text{or} \quad x = -1$$

The derivative is always defined and is zero only at these three points - nowhere else.

Third, we check the sign of y' for x-values between the three places where the slope is zero, $x = -2$, $x = -1/2$, $x = 1/2$, and $x = 2$. Notice that we are computing $f'[x]$, not $f[x]$.

$$\begin{aligned}
&x = -2, y' = 15 \cdot (-2)^2 \cdot [(-2)^2 - 1] = 15 \cdot 4 \cdot [4 - 1] = (+) \text{ slope up}\\
&x = -\tfrac{1}{2}, y' = 15 \cdot (-\tfrac{1}{2})^2 \cdot [(-\tfrac{1}{2})^2 - 1] = 15 \cdot \tfrac{1}{4} \cdot [\tfrac{1}{4} - 1] = (-) \text{ slope down}\\
&x = \tfrac{1}{2}, y' = 15 \cdot (\tfrac{1}{2})^2 \cdot [(\tfrac{1}{2})^2 - 1] = 15 \cdot \tfrac{1}{4} \cdot [\tfrac{1}{4} - 1] = (-) \text{ slope down}\\
&x = 2, y' = 15 \cdot (2)^2 \cdot [(2)^2 - 1] = 15 \cdot 4 \cdot [4 - 1] = (+) \text{ slope up}
\end{aligned}$$

Fourth, we compute the (x, y)-coordinates of several important points. The points with zero slope are important, and we can get a reasonable idea of the shape of the curve with only these three values. Notice that now we are using the formula $y = f[x]$ and not $f'[x]$.

$$x = -1, y = 3 \cdot (-1)^5 - 5 \cdot (-1)^3 = -3 + 5 = 2 \qquad (-1, 2)$$

$$x = 0, y = 3 \cdot (0)^5 - 5 \cdot (0)^3 = 0 \qquad (0, 0)$$

$$x = 1, y = 3 \cdot (1)^5 - 5 \cdot (1)^3 = 3 - 5 = -2 \qquad (1, -2)$$

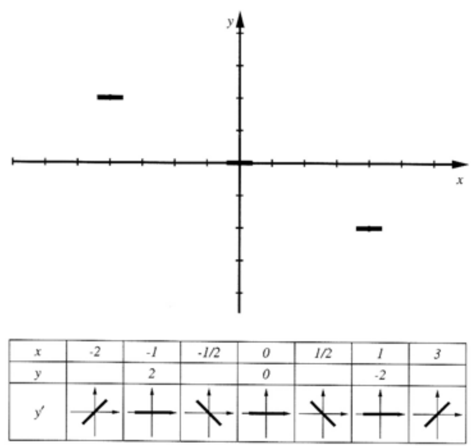


Figure 9.3:7: Slope information for $y = 3x^5 - 5x^3$

Mathematica's version of the graph is shown in Figure 9.3:8.

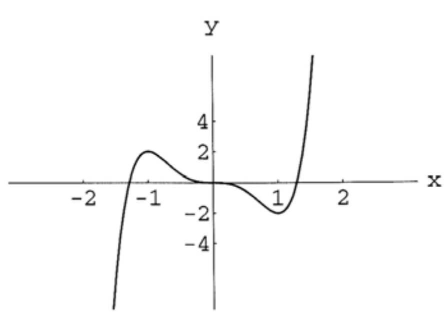


Figure 9.3:8: $y = 3x^5 - 5x^3$

Example 9.3 *Graphing $y = x\,e^{-x}$*

First, use the Product Rule and Chain Rule to compute the derivative

$$y = f[x]\,g[x] \qquad f[x] = x \qquad \tfrac{df}{dx} = 1$$

$$g[x] = e^{-x} \qquad g = e^{u} \qquad u = -x$$

$$\tfrac{dg}{du} = e^{u} \qquad \tfrac{du}{dx} = -1$$

$$\tfrac{dg}{dx} = \tfrac{dg}{du}\,\tfrac{du}{dx}$$

$$= (e^{u})(-1) = -e^{-x}$$

So, the final derivative is

$$\frac{dy}{dx} = \frac{df}{dx}\,g + f\,\frac{dg}{dx} = 1\,e^{-x} - x\,e^{-x} = (1-x)\,e^{-x}$$

This derivative is defined for all real x.

We know that $e^{u} \neq 0$ for any u, so $\frac{dy}{dx} = 0$ only if $1 - x = 0$ or $x = 1$.

Second, we check signs at x values between $-\infty$ and 1 and between 1 and $+\infty$, $x = -1$ satisfies $-\infty < -1 < 1$ and $x = 3$ satisfies $1 < 3 < +\infty$.

$$y'(-1) = (1+1)\,e^{+1} = 2 \cdot e \approx 5.4$$
$$y'(+3) = (1-3)\,e^{-3} = -2\,e^{-3} \approx -0.10$$

The values of the y-coordinate at these points are

$$y(-1) = (-1)\,e^{+1} = -e \approx -2.7$$
$$y(+1) = 1\,e^{-1} = 1/e \approx 0.368$$
$$y(+3) = 3\,e^{-3} = 3/e^{3} \approx 0.15$$

We also know that

$$\lim_{x\to-\infty} x\,e^{-x} = -\infty \quad \& \quad \lim_{x\to\infty} \frac{x}{e^{x}} = 0$$

and that $y = x\,e^{-x} > 0$ for $x > 0$. (See Theorem 8.3.)

All this graphing information is recorded in the table of Figure 9.3:9.

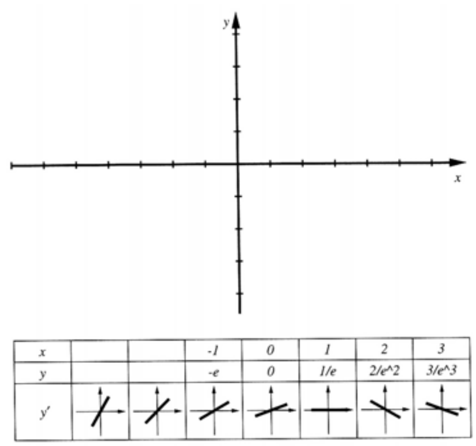


Figure 9.3:9: Slope information for $y = xe^{-x}$

Exercise Set 9.3

The next exercise has a row of microscopic views filled out. Just looking across the y' row we see that the graph goes up - over - down - over -down - over - up. Plot the points given in the x and y row. Add little tangent segments given in the y' row at the (x, y) points. Then fill in a curve that goes through the points and is tangent to the segments.

1. *The table below the axes in Figure 9.3:10 contains x and y coordinates of the point as well as microscopic views of $y = f[x]$ at those dotted points. Sketch the graph.*

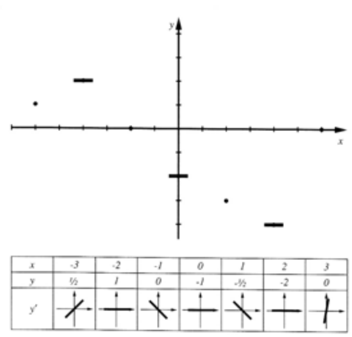


Figure 9.3:10: Points and microscopic views

Reverse the procedure of the last exercise. Look at the next graph, fill out the x and y numbers and make microscopic views of the graph at these points.

2. *Fill out the table in Figure 9.3:11 to correspond to the given graph.*

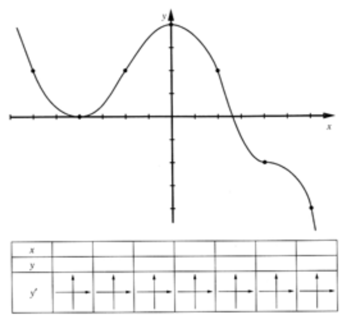


Figure 9.3:11: Look in Your Imaginary Microscope

3. *Graphing with Slopes Drill*
   Use the above first derivative procedure to sketch the graphs of

a) $y = f[x] = 2x^2 - x^4$	*b*) $y = f[x] = x^3 - 3x$
c) $y = f[x] = 6x^5 - 10x^3$	*d*) $y = (x+2)^{\frac{2}{3}}$
e) $y = \mathrm{Sin}[x] + \mathrm{Cos}[x]$	*f*) $y = \mathrm{Sin}[x] \times \mathrm{Cos}[x]$
g) $y = x\ e^x$	*h*) $y = e^{-x^2}$
i) $y = x\ \mathrm{Log}[x]$	*j*) $y = x - \mathrm{Log}[x]$

You may check your graphs with the computer after you sketch by hand. Later, you will need the skills you develop in this exercise together with the computer in order to understand complicated graphs like Planck's Formula in the projects.

9.3.1 The Theorems of Bolzano and Darboux

How do we know that it is sufficient to just check one point between the zeros of $f'[x]$ in the graphing procedure of the last section? This is because derivatives have the property that they cannot change sign without being zero, provided that they are defined on an interval. If $f'[x]$ is not zero in an interval $a < x < b$, then $f'[x]$ cannot change sign. This is taken up in the Mathematical Background Chapter on Bolzano's Theorem, Darboux' Theorem, and the Mean Value Theorem, see http://www.math.uiowa.edu/˜stroyan/InfsmlCalc/FoundationsTOC.htm.

9.4 Bending & the Second Derivative

If the slope gets steeper and steeper, the curve bends up. The derivative $f'[x]$ is just a function and its derivative must be positive if it is increasing, $f''[x] = (+) > 0$. A negative second derivative make the first derivative decrease. If the slope decreases, the curve bends down. These two facts can be summed up in the diagram of Figure 9.4:12.

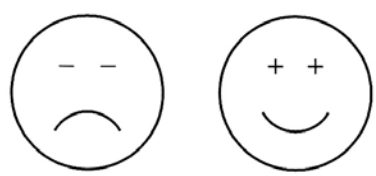
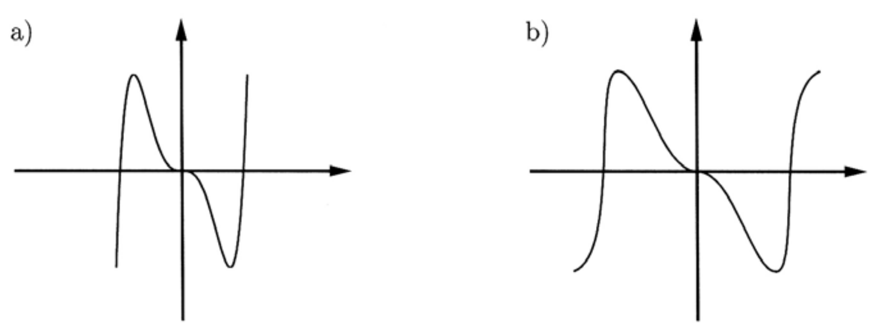


Figure 9.4:12: $f''[x]$ Negative on frown - positive on smile

This section shows you how to use this diagram.

Which is a better graph of graph of $y = 3x^5 - 5x^3$?



The difference between the choices is the bending, not the slope. This information can be obtained by asking whether the first derivative is increasing or decreasing.

Example 9.4 *The Bends of* $y = 3\,x^5 - 5\,x^3$

We computed the slope information of $y = 3\,x^5 - 5\,x^3$ in Example 9.2 above. The derivatives are

$$\begin{aligned} y &= 3\,x^5 - 5\,x^3 \\ y' &= 3 \cdot 5\,x^4 - 5 \cdot 3\,x^2 = 15\,(x^4 - x^2) \\ y'' &= 15\,(4\,x^3 - 2\,x) = 30\,x\,(2\,x^2 - 1) \end{aligned}$$

We already found the slope table and (x, y)-points at the places where the slope is zero. Now we want to find out where the curve bends up (or looks like part of a smile) and where it bends down.

$$\begin{aligned} y'' = 0 \quad &\Leftrightarrow \quad 0 = 30\,x\,(2\,x^2 - 1) \quad \Leftrightarrow \quad x = 0 \quad \text{or} \quad 2\,x^2 - 1 = 0 \\ &\Leftrightarrow \quad x = 0 \quad \text{or} \quad x = \frac{1}{\sqrt{2}} \approx 0.707 \quad \text{or} \quad x = -\frac{1}{\sqrt{2}} \approx -0.707 \end{aligned}$$

The second derivative is always defined and only is zero at these three points.

Now we check the signs of the second derivative at values between the zeros.

$$\begin{array}{lll} x = -1, & y'' = 30 \cdot (-1) \cdot [2(-1)^2 - 1] = -30 \cdot [4 - 1] = (-) & \text{frown} \\ x = -\frac{1}{2}, & y'' = 30 \cdot (-\frac{1}{2}) \cdot [2(-\frac{1}{2})^2 - 1] = -15 \cdot [\frac{1}{2} - 1] = (+) & \text{smile} \\ x = \frac{1}{2}, & y'' = 30 \cdot (\frac{1}{2}) \cdot [2(\frac{1}{2})^2 - 1] = 15 \cdot [\frac{1}{2} - 1] = (-) & \text{frown} \\ x = 1, & y'' = 30 \cdot (1) \cdot [2(1)^2 - 1] = 30 \cdot [4 - 1] = (+) & \text{smile} \end{array}$$

So we see that graph (a) above is a better representation of the curve $y = 3\,x^5 - 5\,x^3$, because the extra bends on graph (b) have the bending sequence smile - frown - smile - frown - smile - frown. We could eliminate graph (b) because the second derivative at a large positive number would need to be negative in order to have the right-most downward (frown) bend shown on graph (b). Similarly, the left-most upward bend would require that the second derivative is positive. Graph (a) above has the (-) - (+) - (-) - (+) or frown - smile - frown - smile sequence of signs to its second derivative.

Sketching curves with both the slope and bend information amounts to filling out the table of Figure 9.4:13 according to the following procedure.

Procedure 9.2 *Plotting with the First and Second Derivatives*

1. (a) Compute $f'[x]$ and find **all** values of x where $f'[x] = 0$ (or $f'[x]$ does not exist). Record these in the microscope row as horizontal line segments (or $*s$ if the derivative does not exist).

   (b) Check the sign (+) or (-) of $f'[x]$ at values between each of the points from the first part. Record $f'[x] = (+) > 0$ as an upward sloping microscope line and $f'[x] = (-) < 0$ as a downward sloping microscope line.

   (c) Compute $f''[x]$ and find **all** values of x where $f''[x] = 0$ (or $f''[x]$ does not exist). Record these in the y'' row of the table as $0s$ (or $*s$ if the derivative does not exist).

   (d) Check the sign (+) or (-) of $f''[x]$ at values between each of the points from the first part. Record $f''[x] = (+) > 0$ as a smile and $f''[x] = (-) < 0$ as a frown.

   (e) Compute a few key (x, y) pairs (using $f[x]$ to find values of y, not $f'[x]$ or $f''[x]$) and record the numbers on the x and y rows. For example, you should at least compute the (x, y) pairs for the x values used in steps (a) and (c).

   (f) Plot the points and mark small tangent lines.

   (g) Connect the points with a curve matching the tangents as you pass through the points and increasing or decreasing and bending according to the table.

Do not start by plotting points. Start with the "shape" information of the slope row and then do the bend row so you first find out which points are interesting to plot.

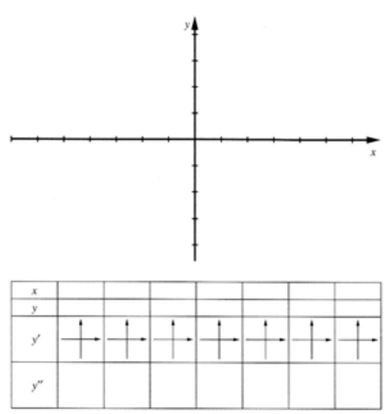


Figure 9.4:13: Blank table for plotting with x, y, y' and y''

Example 9.5 $y = x^3 + x$

The graph of $y = x^3 + x$ illustrates the additional information of the second derivative. We have $\frac{dy}{dx} = 3x^2 + 1$ which is always positive. The first derivative slope information just says the graph is increasing. However, $\frac{d^2y}{dx^2} = 6x$ which is zero at $x = 0$ and only there. When $x < 0$, $\frac{d^2y}{dx^2} < 0$, so the graph bends downward, but slopes upward. The left half of a frown slopes up, but bends down. When $x > 0$, $\frac{d^2y}{dx^2} > 0$, so the graph bends upward and slopes upward. The right half of the smile slopes up, and bends up. Fill out the slope and bend shape tables for this graph which is given next.

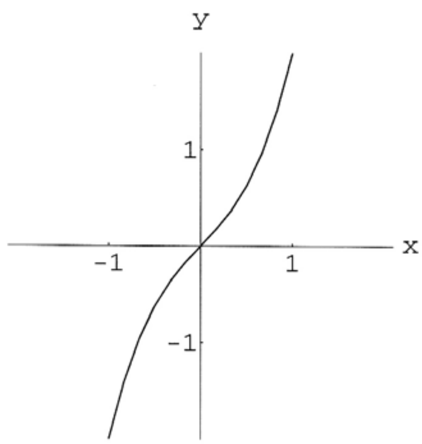


Figure 9.4:14: $y = x^3 + x$

Example 9.6 *Graphing* $y = x\,e^{-x}$ *- (con't).*

We now add the second derivative information to the sketch of the graph of Example 9.3. We know from that example that $\frac{dy}{dx} = (1 - x)\,e^{-x}$, so the second derivative is

$$\frac{d^2y}{dx^2} = (x - 2)\,e^{-x}$$

which is defined everywhere.

The second derivative is zero only if $x = 2$, since $e^u > 0$ for all u.

We check the values $x = 0$ with $-\infty < 0 < 2$ and $x = 3$ with $2 < 3 < +\infty$,

$$y''[0] = -2\,e^0 = -2 \qquad \text{frown}$$
$$y''(3) = (3-2)\,e^{-3} = 1/e^3 > 0 \qquad \text{smile}$$

The slope and bend information we have computed is recorded in Figure 9.4:15.

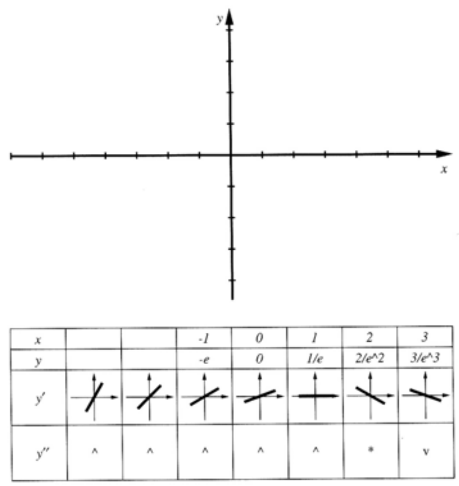


Figure 9.4:15: Slope and bend information for $y = xe^{-x}$

The computer graph is in Figure 9.4:16.

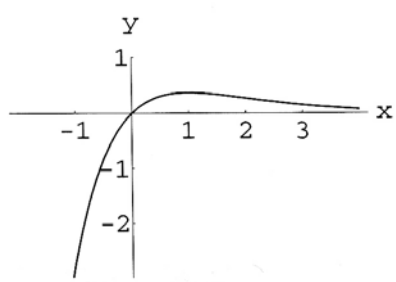


Figure 9.4:16: The computer graph of $y = xe^{-x}$

Example 9.7 *Graph* $y = e^{-x^4}$

Begin with the Chain Rule,

$$y = e^u \qquad u = -x^4$$

$$\frac{dy}{du} = e^u \qquad \frac{du}{dx} = -4x^3$$

so $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -4x^3\, e^{-x^4}$$

We have $\frac{dy}{dx} = 0$ only if $x = 0$ and the derivative is defined everywhere. Check $\frac{dy}{dx}$ at $x = \pm 1$ to see that the slope table is up - over - down.

Now use the Product Rule on $\frac{dy}{dx} = f[x]\, g[x]$,

$$f = -4x^3 \qquad g = e^{-x^4}$$

$$\frac{df}{dx} = -12x^2 \qquad \frac{dg}{dx} = -4x^3 e^{-x^4}$$

so

$$\frac{d^2y}{dx^2} = \frac{df}{dx}\, g + f\, \frac{dg}{dx} = -12x^2\, e^{-x^4} + 16x^6\, e^{-x^4}$$

$$\frac{d^2y}{dx^2} = (16x^6 - 12x^2)\, e^{-x^4}$$

The second derivative is defined for all x and only equals zero when $16x^6 - 12x^2 = 0$. This happens at $x = 0$ and $16x^4 = 12$ or $x = \pm\sqrt[4]{\frac{3}{4}} \approx \pm 0.9306$. Checking values, we see that the bending table is smile - frown - frown - smile.

The limiting values as $x \to \pm\infty$ are simple since $-x^4 < 0$ tends to $-\infty$, $y \to 0$.

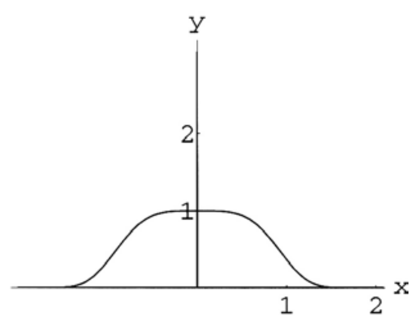


Figure 9.4:17: $y = e^{-x^4}$

Example 9.8 *Graph* $y = x\,\mathrm{Log}[x]$

SOLUTION: The Product Rule makes the derivative

$$\frac{dy}{dx} = \mathrm{Log}[x] + x\frac{1}{x} = 1 + \mathrm{Log}[x]$$

Setting this equal to zero, we find that

$$\frac{dy}{dx} = 0 \quad \Leftrightarrow \quad x = e^{-1} = \frac{1}{e}$$

and the slope table on $(0, \infty)$ is down-over-up. This makes the minimum occur at $x = 1/e$. We also know that $x\,\mathrm{Log}[x]$ is negative for $x < 1$.

The rest of the graphing information is easy to get,

$$\frac{d^2y}{dx^2} = \frac{1}{x} > 0$$

for all $x > 0$, so the curve always bends upward.

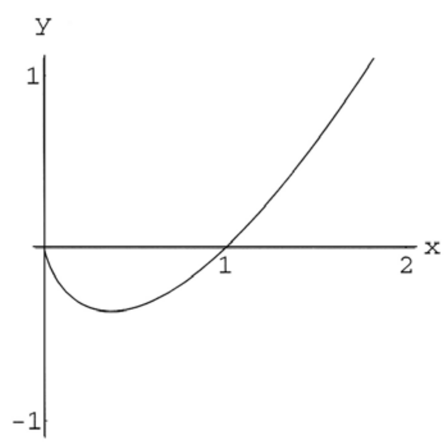


Figure 9.4:18: $y = x\,\mathrm{Log}\,[x]$

When $x = H$ an infinitely large number, $H\,\mathrm{Log}[H]$ is also infinitely large, so

$$\lim_{x\to\infty} x\,\mathrm{Log}[x] = \infty$$

The question is, "What value does $x\,\mathrm{Log}[x]$ increase to as x decreases to zero?"

$$\lim_{x \downarrow 0} x\,\mathrm{Log}[x] = ?$$

To compute the limit at zero, we make a change of variables. Let $z = 1/x$ and rewrite our problem

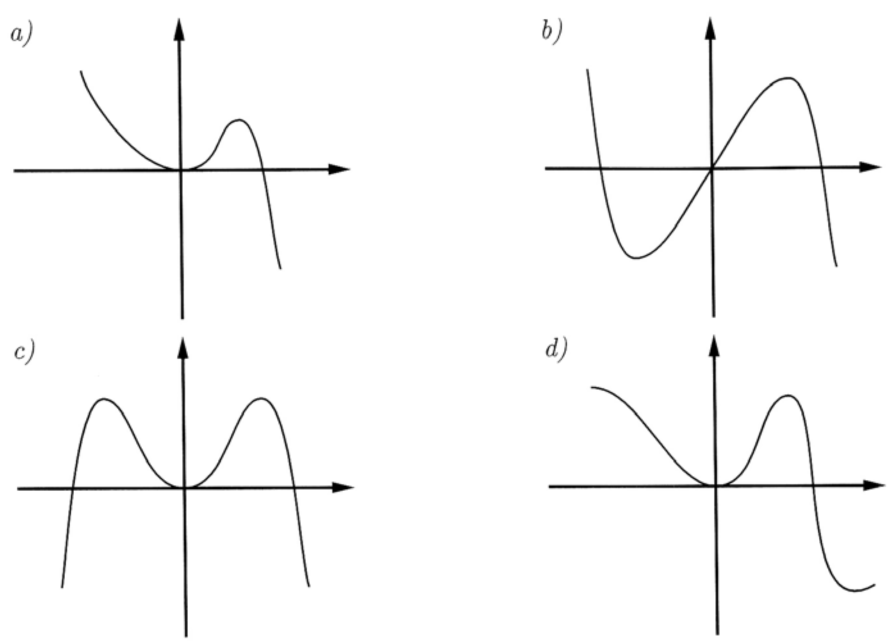
$$\lim_{x \downarrow 0} x\,\mathrm{Log}[x] = \lim_{z \to \infty} \frac{\mathrm{Log}[1/z]}{z} = \lim_{z \to \infty} \frac{-\mathrm{Log}[z]}{z} = 0$$

(See Example 8.7.)

Exercise Set 9.4

Identify bends in the figures of the next exercise and then compute y'' from the given formula to see which graph is right.

1. *Which graph is nearest $y = 4x^2 - \frac{x^5}{5}$?*



2. *Use first and second derivative procedure to sketch graphs of*

   a) $y = f[x] = 2x^2 - x^4$ *b)* $y = f[x] = x^3 - 3x$

   c) $y = f[x] = 6x^5 - 10x^3$ *d)* $y = (x+2)^{\frac{2}{3}}$

   e) $y = \mathrm{Sin}[x] + \mathrm{Cos}[x]$ *f)* $y = \mathrm{Sin}[x] \times \mathrm{Cos}[x]$

   g) $y = x\ e^x$ *h)* $y = e^{-x^2}$

   i) $y = \dfrac{\mathrm{Log}[x]}{x}$ *j)* $y = x - \mathrm{Log}[x]$

   k) $y = \dfrac{1}{1+x^2}$ *l)* $y = \dfrac{1}{1-x^2}$

   Check your graphs with the computer, but remember that these are practice problems for you to work by hand. Later, messy real-world problems (like Planck's Law in the Projects) will require this calculus effort before you start the computer.

Graphical analysis is very useful and often only a rough idea is enough. Here is an example of using one graph to help find another.

Problem 9.1 *Graph* $y = (x^3 - x^2 + 1)^2$*. How does the graph of the simpler equation* $z = x^3 - x^2 + 1$ *help in finding* ***all*** *the places where* $y'(x) = 0$*?*

9.5 Graphing Differential Equations

A differential equation, such as

$$\frac{dy}{dt} = y\,(3-y)$$

can be thought of as a description of the slope of the curve $y = y[t]$*, given that you know* y*. This is enough to sketch a graph.*

Example 9.9 *The Slopes of $y[t]$ When $\frac{dy}{dt} = y\,(3-y)$*

In this case, if we start at $y = 1$ when $t = 0$, the initial slope is $\frac{dy}{dt} = y\,(3-y) = 1(3-1) = 2$. We can begin to sketch the curve by putting our pencil at $(0, 1)$ and moving up along a line of slope 2. After we have sketched a small distance, both t and y will be larger and the slope will change accordingly.

We can move a specific small amount to y_1 and recompute the slope from the differential equation, $y_1\,(3-y_1)$, as we have the computer do in the **SecondSIR** program and the computations on the dead canary, Exercise 4.2.1. The specific amounts are not essential to sketch the curve. Whenever y is between 1 and 3, the slope, $\frac{dy}{dt} = y\,(3-y)$ is positive. This means that as t increases, y increases. However, as we approach $y = 3$ from below, the term $3 - y$ tends to zero and the slope $\frac{dy}{dt} = y\,(3-y)$ also tends to zero. This means that the rate of increase slows down as y approaches 3. If we ever get to $y = 3$, the slope becomes zero and the curve ceases to increase. This rough reasoning gives a sketch of the solution of the differential equation when the solution starts at $y[0] = 1$, as shown as the lowest graph in Figure 9.5:19. It does not require a formula for the solution.

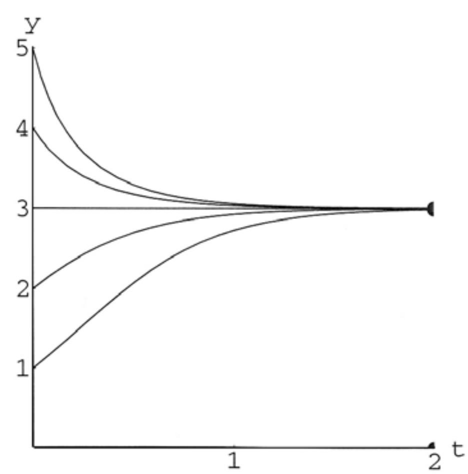


Figure 9.5:19: Solutions to $\frac{dy}{dt} = y\,(3-y)$, starting at $y\,[0] = 1, 2, 3, 4, 5$

This is a graphical version of Euler's Method or the idea of the **SecondSIR** program. Although it is not very accurate, it does give the behavior of solutions. We will return to this kind of curve sketching when we study differential equations later in the course.

Example 9.10 *The Bends of* $y[t]$ *When* $\frac{dy}{dt} = y\,(3-y)$

We can also ask where the concavity of the curve changes. We use the Product Rule and general Chain Rule to compute

$$\begin{aligned}\frac{d}{dt}\frac{dy}{dt} &= \frac{d}{dt}(y\,(3-y))\\ \frac{d^2y}{dt^2} &= \frac{dy}{dt}\,(3-y) + y\,\frac{d(3-y)}{dt}\\ &= \frac{dy}{dt}\,(3-y) - y\,\frac{dy}{dt} = (3-2y)\,\frac{dy}{dt}\end{aligned}$$

Now, substitute the formula for $\frac{dy}{dt} = y\,(3-y)$, to obtain

$$\frac{d^2y}{dt^2} = (3-2y)\,\frac{dy}{dt} = y\,(3-y)(3-2y)$$

The second derivative equals $y\,(3-y)(3-2y)$, so $y''[t] > 0$ for $0 < y < 3/2$ and $y > 3$, whereas $y''[t] < 0$ for $y < 0$ or $\frac{3}{2} < y < 3$. Notice the change of concavity in Figure 9.5:20.

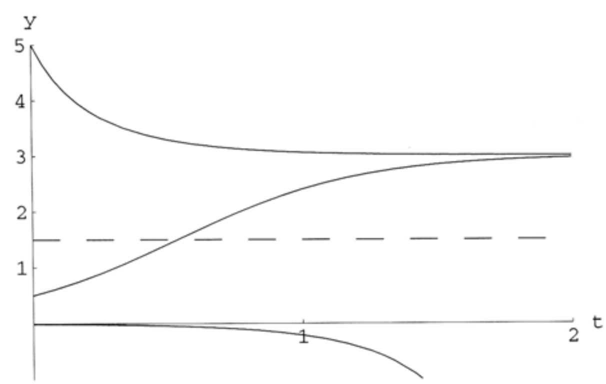


Figure 9.5:20: Concavity of logistic solutions

Exercise Set 9.5

A fundamental "decay law" says that the rate of decrease of a quantity is proportional to the amount that is left. Radioactive substances have this property. You can sketch the amount of such a substance as a function of time without any formula.

1. *Exponential Decay Without Formulas*
   Given that a quantity q satisfies $q[0] = 4$ and $\frac{dq}{dt} = -\frac{1}{2}\, q$, sketch the graph of q vs t. What happens as the quantity as it gets close to $q = 0$? You can find an analytical solution to the conditions $q[0] = 4$ and $\frac{dq}{dt} = -\frac{1}{2}\, q$? (See Chapter 8. This is not needed for the sketch.)

   a) $y[0] = 1$ *b)* $y[0] = 2$ *c)* $y[0] = 6$ *d)* $y[0] = 4$ *e)* $y[0] = 0$

2. *Sketch the solutions of $\frac{dy}{dt} = 10y(4 - y)$ that begin with*

   a) $y[0] = 1$ *b)* $y[0] = 2$ *c)* $y[0] = 6$ *d)* $y[0] = 4$ *e)* $y[0] = 0$

   Euler's approximation uses the tangent line for a small step and then recomputes the slope. The increment equation says, $y[t + \delta t] = y[t] + y'[t] \cdot \delta t + \varepsilon \cdot \delta t$, so $y[t + \delta t] \approx y[t] + y'[t] \cdot \delta t$. If $y[t]$ satisfies $y[0] = y_0$ and $y'[t] = f[y[t]]$, Euler's approximation is given recursively by

$$y_{approx}[0] = y_0$$
$$y_{approx}[t + \delta t] = y_{approx}[t] + f[y_{approx}[t]] \cdot \delta t$$

   for $t = 0, \delta t, 2\delta t, 3\delta t, \ldots$. Notice that the equation of the tangent at a fixed point $y_1 = y[t]$ with slope $m = y'[t]$ is $dy = m \cdot dt$ in local coordinates, where $dy = y - y_1$. So $y = y_1 + m \cdot \delta t$ is the point on the tangent line of slope m at time δt past t.

$$y = y_1 + m \cdot \delta t$$
$$y[t + \delta t] = y[t] + y'[t] \cdot \delta t$$

   If the second derivative $y''[t]$ is positive and $y'[t]$ is increasing in y, Euler's approximation is always low. We can see this just from the relation between the graph and the derivatives.

3. *Suppose $y[0] = 1/4$ and $\frac{dy}{dt} = \sqrt{y}$. Why is the tangent line to the exact solution $y = y[t]$ below the curve at $t = 0$? If Euler's approximate solution $y_{approx}[\delta t]$ is low at the first step, why is the slope $f[y_{approx}[\delta t]]$ below the slope of $y[t]$ at $t = \delta t$? (HINT: Compute $\frac{d^2y}{dt^2} = \frac{1}{2}$ using the Chain Rule. Where does the tangent lie in relation to a curve with a positive second derivative? Draw a tangent on a smiley face.) Explain why the Euler approximation is always below the true solution $y[t]$.*

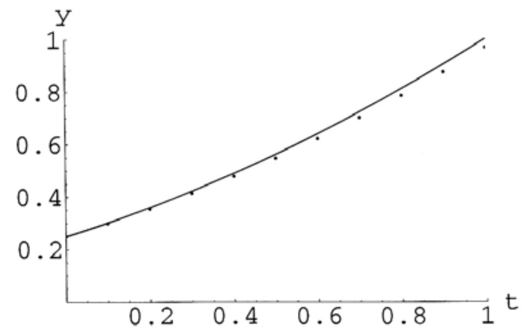


Figure 9.5:21: Euler's approximation to $y[0] = 1/4$ and $y'[t] = \sqrt{y[t]}$

4. *Solutions to the S-I-S Equations*
   An S-I-S disease is one that does not confer immunity; you are either susceptible or infectious. If s is the fraction of the population that is susceptible, the spread of the disease in time t is given by
   $$\frac{ds}{dt} = b(1-s)(1-c\,s)$$
   where b is the reciprocal of the infectious period and c is the contact number. Assume that $c > 1$ (so that each infectious person contacts more than one other person) and sketch the graphs of the solutions for various initial conditions between 0 and 1. (HINTS: Show that s has a positive derivative for $0 < s < 1/c$ and a negative derivative for $1/c < s < 1$. The second derivative $\frac{d^2s}{dt^2} = b^2(s-1)(1-cs)((1+c)-2cs)$, so concavity changes at the average of 1 and $1/c$.)

9.6 Projects

9.6.1 Planck's Formula & Wein's Law

Planck won a Nobel prize for a formula that tells the intensity of radiation as a function of temperature. His formula predicts the empirically observed law of radiation discovered earlier by Wein. The peak in Planck's formula gives Wein's law. This was an important early discovery in quantum mechanics because classical thermodynamics does not predict Wein's Law.

This project shows that graphing alone is not enough to find the peak but that calculus and the computer together make the story clear.

9.6.2 Algebraic Formulations of Increasing and Bending

The project on “Taylor’s Formula,” can be used to give algebraic proofs of the meaning of the slope and bending icons used in graphing.

9.6.3 Bolzano, Darboux, and the Mean Value Theorem

The Mathematical Background chapter on these three theorems provides complete justification for our simple method of finding the slope and bend tables.

9.6.4 Horizontal and Vertical Asymptotes

The limits at infinity that we computed in some of our graphs show that the graph is tending toward a line called an “asymptote.” This project explores this topic in greater detail.