26. Compact Spaces

A family of sets, $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$, **covers** the set S if

$$S \subset \cup_{\alpha \in A} F_\alpha$$

$\mathcal{F}$ is said to be a cover of S.

$\mathcal{F}_1 = \{\ (x-1, x+1) \mid x \in S\}$ is a cover of S since

$$S \subset \cup_{x \in S}(x-1, x+1)$$

If $S \neq \emptyset$, take $x_0 \in S$
$\mathcal{F}_2 = \{\ (x_0 - r, x_0 + r) \mid r > 0\}$ is a cover of S since

$$S \subset \cup_{r>0}(x_0 - r, x_0 + r)$$

$\mathcal{F}'$ is a **subcover** of $\mathcal{F}$ if $\mathcal{F}' \subset \mathcal{F}$ and $\mathcal{F}'$ covers S.

$\mathcal{F}'$ is a **finite subcover** of $\mathcal{F}$ (or a finite subfamily of $\mathcal{F}$) if $\mathcal{F}'$ is a subcover of $\mathcal{F}$ and $\mathcal{F}'$ is finite.

$\mathcal{F} = \{\ (\frac{1}{n}, 1) \mid n = 1, 2, 3, ...\}$ is a cover of $(0,1)$ since $(0,1) \subset \cup_{n=1}^{\infty}(\frac{1}{n}, 1)$.

$\mathcal{F}' = \{\ (\frac{1}{n}, 1) \mid n = 5, 6, 7, ...\}$ is a subcover of $\mathcal{F}$ since $\mathcal{F}' \subset \mathcal{F}$ and $(0,1) \subset \cup_{n=5}^{\infty}(\frac{1}{n}, 1)$.

Does there exist a finite subcover?

Defn: A family of sets, $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$, is an **open cover** of S if $\mathcal{F}$ covers S and if F_α is open for all $\alpha \in A$.

Defn: A space S is **compact** if every open cover of S has a finite subcover.

Lemma: If X is finite, then X is compact.

Lemma 26.1: Let Y be a subspace of X. Every cover of Y consisting of open sets in X has a finite subcover if and only if every cover of Y consisting of open sets in Y has a finite subcover

Thm 26.2: Every closed subspace of a compact space is compact.

Lemma 26.4: If Y is a compact subspace of the Hausdorff space X, and $x_0 \notin Y$, then there exist disjoint open sets U and V of X such that $x_0 \in U$ and $Y \in V$.

Thm 26.3: Every compact subspace of a Hausdorff space is closed.

Thm 26.5: The image of a compact space under a continuous map is compact.

Thm 26.6: If $f : X \to Y$ is continuous and a bijection and if X is compact and Y is Hausdorff, then f is a homeomorphism.

Note: $f : [0,1) \to \{(x,y) \mid x^2 + y^2 = 1\}$, $f(x) = e^{2\pi ix}$ is continuous and a bijection, but f^{-1} is NOT continuous.

Note: $f : \{1,2\} \to \{1,2\}$, $f(n) = n$ is a bijection. $X = \{1,2\}$ is compact since X is finite.

If X has the ________________________ topology and

Y has the ________________________ topology, then

f is continuous, but not a homeomorphism.

$Y = \{1,2\}$ is not ________________________.

Lemma 26.8 (The tube lemma). Suppose N is an open set in $X \times Y$ where Y is compact. If there exists an $x_0 \in X$ such that $x_0 \times Y \subset N$, then there exists an open set W such that $x_0 \in W$ and $W \times Y \subset N$.

Thm 26.7: The product of finitely many compact spaces is compact.

Thm 37.3 (Tychonoff theorem). An arbitrary product of compact spaces is compact in the product topology.

Defn: A collection C is said to have the **finite intersection property** if for every finite subcollection $\{C_1, ..., C_n\} \subset \mathcal{C}$, $\cap_{i=1}^n C_i \neq \emptyset$.

Example 1: $\{(-n, n) \mid n = 1, 2, 3, ...\}$ has/does not have finite intersection property.

Example 2: $\{(n, n+2) \mid n \in \mathcal{Z}\}$ has/does not have finite intersection property.

Example 3: $\{(0, \frac{1}{n}) \mid n = 1, 2, 3, ...\}$ has/does not have finite intersection property.

Thm 26.9: X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, $\cap_{C \in \mathcal{C}} C \neq \emptyset$.