Let $\mathcal{A}$ be a collection of subsets of X. A collection $\mathcal{B}$ of subsets of X is a *refinement* of $\mathcal{A}$ if for all $B \in \mathcal{B}$, there exists $A \in \mathcal{B}$ such that $B \subset A$.

If the elements of $\mathcal{B}$ are open, $\mathcal{B}$ is an *open refinement* of $\mathcal{A}$. If the elements of $\mathcal{B}$ are closed, $\mathcal{B}$ is a *closed refinement* of $\mathcal{A}$. $\blacksquare$

Defn: X is *paracompact* if every open covering of X has a locally finite open refinement that covers X

Lemma 39.2 + Thm 41.4: metrizable implies paracompact.

A collection $\mathcal{A}$ of subsets of X is *countably locally finite* if $\mathcal{A}$ can be written as a countable union of collections $\mathcal{A}_n$, each of which is locally finite.

Ex: $\mathcal{D} = \{(-n, n) \mid n \in \mathbf{Z}_+\}$ is countable locally finite.
Let $\mathcal{D}_k = \{(-n, n) \mid n \in [k, k+2]\}$, $k \in 2\mathbf{Z}_+$.
Note $\mathcal{D} = \cup_{k \in 2\mathbf{Z}_+} \mathcal{D}_k$ and $\mathcal{D}_k$ is locally finite since it's finite.

A simply ordered set X is *well ordered* if every nonempty subset of X has a smallest element (ie $A \subset X$, $A \neq \emptyset$ implies $min(A)$ exists and $min(A) \in A$).

Ex: $\mathbf{Z}$ is NOT well-ordered.
Ex: $\mathbf{Z}_+$ is well-ordered
Ex: $\mathbf{R}_+$ is NOT well-ordered.

The Well-ordering theorem: If X is a set, there exists an order relation on X that is well-ordered.

43: Complete Metric Spaces:

In this section, (X, d) is a metric space.

Defn: x_n is Cauchy if for all $\epsilon > 0$, there exists an N such that for all $n, m > N$, $d(x_n, x_m) < \epsilon$.

Defn: (X, d) is complete if every Cauchy sequence in X converges in X.

Note $\mathbf{R}$ is complete, but $(0, 1)$ is not complete. Hence completeness is NOT a topological property.

Suppose $f : (X, d) \to (Y, D)$ is continuous and bijective.
Is $\rho_X(x_1, x_2) = D(f(x_1), f(x_2))$ a metric on X?
Is $\rho_Y(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$ a metric on Y?

Lemma: convergent implies Cauchy

Lemma: (X, d) complete, A closed in X implies (A, d) is complete.

Lemma: (X, d) complete if and only if $(X, \overline{d})$ is complete where $\overline{d}(x, y) = min\{d(x, y), 1\}$.

Lemma 43.1: (X, d) is complete if every Cauchy sequence has a convergent subsequence.

Lemma: A Cauchy sequence is bounded.

Thm 28.2: If X is metrizable, then TFAE:
(1) X is compact.
(2) X is limit point compact.
(3) X is sequentially compact.

Thm 43.2: R^k is complete in both the euclidean metric d or the square metric ρ.

Lemma 43.3: $\mathbf{x_n} \to \mathbf{x}$ in ΠX_α if and only if

$$\pi_\alpha(\mathbf{x_n}) \to \pi_\alpha(\mathbf{x})$$

Thm 43.4: $\mathcal{R}^\omega$ is complete with respect to

$$D(\mathbf{x}, \mathbf{y}) = sup\{\frac{\overline{d}(x_i, y_i)}{i}\}$$

Recall $Y^J = \{(y_\alpha)_{\alpha \in J}\} = \{f : J \to Y\}$ where $f(\alpha) = y_\alpha$

Thus, $\overline{\rho}(\mathbf{x}, \mathbf{y}) = sup\{\overline{d}(x_\alpha, y_\alpha)\}$ is the same as

$$\overline{\rho}(f, g) = sup\{\overline{d}(f(\alpha), g(\alpha))\}$$

Thm 43.5: (Y, d) complete implies Y^J is complete with respect to the uniform metric.

Defn: The fn $f : X \to Y$ is bounded if $f(X)$ is bounded [there exists $B(x_0, r)$ such that $f(X) \subset B(x_0, r)$].

Defn: $\mathcal{B}(X, Y) = \{f : J \to Y \mid f \text{ bounded }\}$.

If X is a topological space, define

$$\mathcal{C}(X, Y) = \{f : J \to Y \mid f \text{ continuous }\}.$$

Thm 43.6: $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ are closed subsets of Y^X under the uniform topology. Thus if Y is complete in uniform metric, $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ are complete.

Defn: The sup metric, $\rho(f, g) = sup\{d(f(\alpha), g(\alpha)\}$, is a metric on $\mathcal{B}(X, Y)$

Note: $\overline{\rho}(f, g) = \min\{\rho(f, g), 1\}$.

Note: If X compact, $Y^X = \mathcal{B}(X, Y)$

Thm 43.7: There is an isometric imbedding of (X, d) into a complete metric space.

Defn: If $h : X \to Y$ is an isometric imbedding of metric space X into complete metric space Y, then $\overline{h(X)}$ is a complete metric space called the completion of X.

The completion of X is uniquely determined up to isometry. $\blacksquare$

recommended HW 43: 8, 10

44: There exists a continuous surjective function

$$f : [0, 1] \to [0, 1] \times [0, 1].$$