Ph.D. Qualifying Exam in Topology
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Instructions. Do eight problems, four from each part. Some problems may require ideas from both semesters 22M:132-22M:133, and some problems may go beyond what was covered in the course. This is a closed book examination. You should have no books or papers of your own. Please do your work on the paper provided. Clearly number your pages to correspond with the problem you are working. When you start a new problem, start a new page; and please write only on one side of the paper.
You may use "big theorems" provided that the point of the problem is not the proof of the theorem.

Always justify your answers unless explicitly instructed otherwise.

NAME (print) ________________________________

NAME (signature) ________________________________

Please indicate here which eight problems you want to have graded:

A1 A2 A3 A4 A5 A6

B1 B2 B3 B4 B5 B6

Notation:
$\mathbb{R}^n$ is Euclidean n-space, with the usual topology and differentiable structure (unless a different topology or differentiable structure is specified in a particular problem).

$\mathbb{Q}$ denotes the set of rationals, $\mathbb{Z}$ the set of integers.

S^n is the n-sphere, the set of points distance one from the origin in $\mathbb{R}^{n+1}$, with the subspace topology, and with the usual differentiable structure.

Part A

A: Problem 1. *Here are some very basic theorems. Your job is to show that the various hypotheses are necessary.*

(i) Theorem: If $f : X \to \mathbb{R}^1$ is continuous, and X is compact, then f attains (i.e. achieves) a maximum on X.

Show the theorem is false if we assume f is continuous but omit the hypothesis that X is compact.

Show the theorem is false if we assume X is compact but omit the hypothesis that f is continuous.

(ii) Theorem: If $f : X \to Y$ is continuous, and X is connected, then $f(X)$ is connected.

Show the theorem is false if we assume f is continuous but omit the hypothesis that X is connected.

Show the theorem is false if we assume X is connected but omit the hypothesis that f is continuous.

A: Problem 2. *In this problem, we define a topology for the set $\mathbb{R}^1$ that is different from the usual topology.*

For each $x \in \mathbb{R}^1$, and each real number $\epsilon > 0$, let
$V(x, \epsilon) = \{x\} \cup \{q \in \mathbb{Q} : \|x - q\| < \epsilon\}$.

Let $\mathcal{B}$ = the set of all $V(x, \epsilon)$, i.e.

$$\mathcal{B} = \{V(x, \epsilon) : x \in \mathbb{R}, \epsilon > 0\}.$$

The set $\mathcal{B}$ is a basis for a topology, $\mathcal{T}$, on $\mathbb{R}^1$.

PROVE: $\mathcal{T}$ is strictly finer *than the standard topology on $\mathbb{R}^1$.*

A: Problem 3. *Prove the following theorem, or give (and justify) a counterexample:*

If $X_1, X_2, \ldots$ is a countable collection of separable spaces, then the product

$$W = \prod_{n=1}^{\infty} X_n$$

(with the product topology) is separable.

A: Problem 4. *Suppose $(X, \mathcal{T})$ is a topological space where the set X is countable and the topology $\mathcal{T}$ is Hausdorff. Suppose further that $(X, \mathcal{T})$ is " perfect", that is each point of X is an accumulation point (i.e a limit point) of X.*
Prove $(X, \mathcal{T})$ cannot be compact.

Hint: Use an appropriate version of the Baire Category Theorem.

A: Problem 5. *Suppose X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous surjective map. Let Z be the quotient space of X, where the equivalence classes are the point-inverses under f. That is,*

$$Z = \{f^{-1}(y) | y \in Y\}$$

and Z has the quotient topology.

Prove Z and Y are homeomorphic.

A: Problem 6.

(i) *(1/4 credit) Define "regular", "Lindelöf", "normal".*
(ii) *(3/4 credit) Prove: If X is regular and has the Lindelöff property, then X is normal.*

Part B

B: Problem 1. *Two pre-atlases on an m-manifold are equivalent if their union is a pre-atlas. Show that this is an equivalence relation and that each equivalence class contains a unique complete (i.e., maximal) atlas.*

B: Problem 2. *Prove that* S^n *is a regular submanifold of* $\mathbb{R}^{n+1}$*. Give an example of a subset of* $\mathbb{R}^2$ *which is a manifold, but not a regular submanifold of* $\mathbb{R}^2$*.*

B: Problem 3. *Suppose* $f : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{m+1} - \{0\}$ *is smooth and suppose for some integer* k*,* $f(c\mathbf{x}) = c^k f(\mathbf{x})$ *for all* $c \in \mathbb{R} - \{0\}$ *and* $\mathbf{x} \in \mathbb{R}^{n+1} - \{0\}$*. Show* $F : \mathbb{RP}^n \to \mathbb{RP}^m$ *defined by* $F([x]) = [f(x)]$ *is well-defined and smooth.*

B: Problem 4. *Let* M *be a manifold. Define the tangent bundle* TM*. Give a basis for the topology on* TM*. Give a pre-atlas for* TM*. Show that* $\pi : TM \to M$*,* $\pi(p, v) = p$*, where* $p \in M$ *and* $v \in T_P(M)$*, is smooth.*

B: Problem 5. *Let* $f : \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*
$f(n, (x, y)) = (x + n, (-1)^n y)$*.*
(i) Show that f *is a smooth group action. Show that this action is free and properly discontinuous.*
(ii) Let $M = \mathbb{R}^2/\mathbb{Z}$ *(in the sense of the action given in part (i) above). Show that the projection map* $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$*,* $\pi_1((x, y)) = x$ *induces a smooth map,* $p : M \to S^1$*.*

B: Problem 6. *Let* M *be an* $m-$*dimensional manifold. Define* $\pi : TM \to M$ *by* $\pi(y, v) = y$ *where* $y \in M$ *and* $v \in T_y(M)$*. Define* $p_1 : M \times \mathbb{R}^m \to M$ *by* $p_1(y, \mathbf{x}) = y$ *where* $y \in M$ *and* $\mathbf{x} \in \mathbb{R}^m$*. Suppose there is a smooth map* t*:* $TM \to M \times \mathbb{R}^m$ *so that* $p_1 \circ t = \pi$*, and so that* t*:* $\pi^{-1}(y) = T_yM \to \{y\} \times \mathbb{R}^m$ *is a linear isomorphism for all* y*. Prove that* TM *admits* m *linearly independent vector fields.*