Defn: G is a *topological group* if

1.) $(G, *)$ is a group

2.) G is a topological space.

3.) $* : G \times G \to G$, $*(g_1, g_2) = g_1 * g_2$, and
$In : G \to G$, $In(g) = g^{-1}$ are both continuous functions.

Defn: G is a *Lie group* if

1.) G is a topological group

2.) G is a smooth manifold.

3.) $*$ and In are smooth functions.

Ex: $(\mathbf{R}, +)$, $(\mathbf{R} - \{0\}, \cdot)$, $(\mathbf{C} - \{0\}, \cdot)$, $(S^1, \cdot)$ where $S^1 \subset \mathbf{C}$, $(\mathbf{Z}, +)$, $(\mathbf{Z}_p, +)$, $(Gl(n, \mathbf{R}), \textit{matrix multiplication})$ are Lie groups. For G_1, G_2 lie groups, $G_1 \times G_2$ is a lie group.

Defn: G = group, X = set. G *acts on* X (on the left) if $\exists \sigma : G \times X \to X$ such that

1.) $\sigma(e, x) = x \quad \forall x \in X$

2.) $\sigma(g_1, \sigma(g_2, x)) = \sigma(g_1 g_2, x)$

Notation: $\sigma(g, x) = gx$.
Thus 1) $ex = x$; 2) $g_1(g_2 x) = (g_1 g_2)(x)$.

If G is a topological group and X is a topological space, then we require σ to be continuous.

If G is a Lie group and X is a smooth manifold, then we require σ to be smooth.

Defn: The *orbit* of $x \in X$ =
$G(x) = \{y \in X \mid \exists g \text{ such that } y = gx\}$

Note:
1.) $x \in G(x)$
2.) If $G(x) \cap G(y) \neq \emptyset$, then $G(x) = G(y)$

Thus we can use an action of G to partition X into disjoint subsets.

Hence the action of G on X can be used to define an equivalence relation on X: $x \sim y$ iff $y \in G(x)$ iff $\exists g$ such that $y = gx$.

$$X/G = X/\sim.$$

If X is a topological space, then $X/G = X/\sim$ is a topological space with the quotient topology.

When is $X/G = X/\sim$ a manifold?

Ex: $G = (\mathbf{Z}, +)$, $M = \mathbf{R}$, $\sigma(n, x) = n + x$.

$M/G =$

Ex: $G = (\mathbf{Z} \times \mathbf{Z}, +)$, $M = \mathbf{R}^2$,
$\sigma((n, m), (x, y)) = (n + x, m + y)$.

$M/G =$

Ex: $G = (\mathbf{Z_2}, +)$, $M = S^n$, $\sigma(0, x) = x$, $\sigma(1, x) = -x$, .

$M/G =$

Defn: The action of G on X is *free* if $gx = x$ implies $g = e$.

Thm 1.3.9: If M is a smooth n-manifold, and G is a finite Lie group acting freely on M, then M/G is a smooth n-manifold. Also, $p : M \to M/G$ is smooth.

Cor:

Defn: G is a *discrete group* if

0.) G is a group.

1.) G is countable

2.) G has the discrete topology

Note a discrete group is a Lie group.

Defn: The action of G on M is *properly discontinuous* if $\forall x \in M$, $\exists U^{open}$ such that $x \in U$ and $U \cap gU = \emptyset \ \ \forall g \in G$.

Ex: $(\mathbf{Z}, +)$ acting on $\mathbf{R}^1$ where $\sigma(n, x) = n + x$.

Thm 1.3.2: M smooth n-manifold, G discrete group acting properly discontinuously on M implies M/G is a smooth n-manifold. Also, $p : M \to M/G$ is smooth.