Thm: Let $T : V \to W$ be a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$

Pf: $T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$

Thm: Let A be an $m \times n$ matrix. Then the function

$$T : \mathbf{R}^n \to \mathbf{R}^m$$
$$T(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation.

Thm: If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = [T(\mathbf{e_1})...T(\mathbf{e_n})]$$

Ex: If $T : \mathbf{R}^2 \to \mathbf{R}^2$ and

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$, then

$$T\begin{pmatrix}x\\y\end{pmatrix} = xT\begin{pmatrix}1\\0\end{pmatrix} + yT\begin{pmatrix}0\\1\end{pmatrix} = x\begin{pmatrix}2\\3\end{pmatrix} + y\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \blacksquare$$

Change of basis:

Suppose $\mathcal{S} = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$

Suppose $\mathcal{B} = \{\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}\}$

Defn: $\begin{pmatrix}x\\y\end{pmatrix}_{\mathcal{B}} = x\mathbf{b_1} + y\mathbf{b_2}$

Thus $\begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{B}} = \begin{pmatrix}2\\3\end{pmatrix}_{\mathcal{S}}$ and $\begin{pmatrix}0\\1\end{pmatrix}_{\mathcal{B}} = \begin{pmatrix}1\\4\end{pmatrix}_{\mathcal{S}}$

$$\begin{pmatrix}x\\y\end{pmatrix}_{\mathcal{B}} = x\begin{pmatrix}2\\3\end{pmatrix}_{\mathcal{S}} + y\begin{pmatrix}1\\4\end{pmatrix}_{\mathcal{S}} = \begin{pmatrix}2x+y\\3x+4y\end{pmatrix}_{\mathcal{S}}$$

$$\begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}_{\mathcal{B}} = \begin{pmatrix}u\\v\end{pmatrix}_{\mathcal{S}}$$

Suppose $\mathcal{C} = \{\begin{pmatrix}5\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\}$

$$\begin{pmatrix}5 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}w\\z\end{pmatrix}_{\mathcal{C}} = \begin{pmatrix}u\\v\end{pmatrix}_{\mathcal{S}}$$

$$\begin{pmatrix}1 & -2\\-2 & 5\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}_{\mathcal{S}} = \begin{pmatrix}w\\z\end{pmatrix}_{\mathcal{C}}$$

$$\begin{pmatrix}1 & -2\\-2 & 5\end{pmatrix}\begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}_{\mathcal{B}} = \begin{pmatrix}1 & -2\\-2 & 5\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}_{\mathcal{S}} = \begin{pmatrix}w\\z\end{pmatrix}_{\mathcal{C}}$$

$$\begin{pmatrix}-4 & -7\\11 & 18\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}_{\mathcal{B}} = \begin{pmatrix}w\\z\end{pmatrix}_{\mathcal{C}}$$

Note:
$\begin{pmatrix}-4 & -7\\11 & 18\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}_{\mathcal{B}} = \begin{pmatrix}-4\\11\end{pmatrix}_{\mathcal{C}}$ and $\begin{pmatrix}-4 & -7\\11 & 18\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}_{\mathcal{B}} = \begin{pmatrix}-7\\18\end{pmatrix}_{\mathcal{C}} \blacksquare$

Basis for $T_{(1,0)}(S^1)$ (polar coordinates):

$\phi : S^1 - \{e^{i\pi}\} \to (-\pi, \pi), \quad \phi(e^{i\theta}) = \theta$

Note: $\phi(e^{i0}) = \mathbf{0}$,

The *standard basis* for $T_{(1,0)}(S^1)$ w.r.t. $(U, \phi) = \{D_\alpha\}$ where

$\alpha : (-\epsilon, \epsilon) \to M,\ \alpha(t) = \phi^{-1}(t) = e^{it} = (1, t)_p = (cost, sint)_E$ for some $\epsilon > 0$.

$G((1,0)) = \{f^{smooth} : U \subset S^1 \to \mathbf{R}\}$

$D_\alpha : G((1,0)) \to \mathbf{R}$

$D_\alpha(g) = \frac{d(g \circ \alpha)}{dt}|_{t=0} = \frac{d(g(\phi^{-1}(t)))}{dt}|_{t=0} = \frac{d(g(cos(t), sin(t)))}{dt}|_{t=0}$

If $v \in T_p(M)$, then $v = cD_\alpha$ where $c = v([\pi_1 \circ \phi]) = v([\phi])$

Basis for $T_{(1,0)}(S^1)$ (projection):

$\psi_{yp} : \{\theta \mid -\pi/2 < \theta < \pi/2\} \to (-1, 1), \quad \phi(x, y) = y$

The *standard basis* for $T_{(1,0)}(S^1)$ w.r.t. $(U', \phi) = \{D_\beta\}$ where

$\beta : (-\epsilon, \epsilon) \to M,\ \beta(t) = \psi_{yp}^{-1}(t) = (x, t) = (\sqrt{1 - t^2}, t)$
for some $\epsilon > 0$.

$D_\beta(g) = \frac{d(g \circ \alpha)}{dt}|_{t=0} = \frac{d(g(\psi_{yp}^{-1}(t)))}{dt}|_{t=0} = \frac{d(g(\sqrt{1-t^2}, t))}{dt}|_{t=0}$

If $v \in T_p(M)$, then $v = cD_\beta$ where $c = v([\pi_1 \circ \psi_{yp}]) = v([\psi_{yp}])$

$D_\alpha, D_\beta \in T_p(M)$,

$D_\alpha \in T_p(M)$ implies $D_\alpha = cD_\beta$ where $c = D_\alpha([\psi_{yp}])$

$D_\beta \in T_p(M)$ implies $D_\beta = cD_\alpha$ where $c = D_\beta([\pi_1 \circ \phi]) = D_\beta([\phi])$

$c = D_\beta(\phi) = \frac{d(\phi(\sqrt{1-t^2}, t))}{dt}|_{t=0} = \frac{d(sin^{-1}(t))}{dt}|_{t=0} = \frac{1}{\sqrt{1-t^2}}|_{t=0} = 1$

Thus if $Ax_{polar} = x_{proj}$, then $A =$

Suppose $Bx_{\mathcal{B}} = x_{\mathcal{S}}$ and $Cx_{\mathcal{C}} = x_{\mathcal{D}}$

If $Tx_{\mathcal{B}} = y_{\mathcal{C}}$, then $TB^{-1}Bx_{\mathcal{B}} = C^{-1}Cy_{\mathcal{C}}$

$CTB^{-1}(Bx_{\mathcal{B}}) = Cy_{\mathcal{C}}$

$CTB^{-1}x_{\mathcal{S}} = y_{\mathcal{D}}$

Suppose $\mathcal{C} = \{3D_{\beta}\}$

Then $[3]x_{\mathcal{C}} = x_{proj}$

Suppose the derivative of $f : S^1 \to S^1$ with respect to ψ_{yp} is Df

Then $Df\mathbf{v}_{proj} = \mathbf{w}_{proj}$, then $Df[1/3][3]\mathbf{v}_{proj} = \mathbf{w}_{proj}$.

Hence $(1/3)Df\mathbf{v}_{\mathcal{C}} = \mathbf{w}_{proj}$.