2.5 Combinations of Multisets

Thm 2.5.1 Let $S = \{\infty \cdot a_1, ..., \infty \cdot a_k\}$. Then the number of r-combinations of S is

Proof: The number of r-combinations of S

= the number of integral solutions to the equation

$$x_1 + x_2 + ... + x_k = r \qquad (*)$$

where $x_i \geq 0\ \forall i$ (and where x_i = the number of a_i's chosen for an $r-$combination).

= the number of permutations of $\{r \cdot 1, (k-1) \cdot +\}$ by the following:

Suppose $(c_1, c_2, ..., c_k)$ is a solution to (*). This corresponds to the permutation $11...1 + 1..1 + ... + 11..1$,

where there are $k-1$ +'s and c_1 1's before the first +, c_i 1's between the (i-1)th and ith +'s for $i = 2, ..., k-1$, and c_k 1's after the last +. Since $c_1 + c_2 + ... + c_k = r$, there are r 1's, and thus $11...1 + 1..1 + ... + 11..1$ is a permutations of $\{r \cdot 1, (k-1) \cdot +\}$.

A permutation of $\{r \cdot 1, (k-1) \cdot +\}$ corresponds to a solution $(c_1, c_2, ..., c_k)$ of (*) where c_1 = the number of 1's before the first +, c_i = the number of 1's between the (i-1)th and ith +'s for $i = 2, ..., k-1$, and c_k = the number of 1's after the last +. Since there are r 1's, $c_1 + c_2 + ... + c_k = r$.

The number of permutations of $\{r \cdot 1, (k-1) \cdot +\}$ is

Corollary: Let $S = \{r \cdot a_1, ..., r \cdot a_k\}$. Then the number of r-combinations of S is

Proof:

Some examples

$S = \{\infty \cdot a_1, \infty \cdot a_2, ..., \infty \cdot a_5\}$.

Then a 4-combination of S is $\{a_3, a_3, a_3, a_5\}$

Suppose $x_1 + x_2 + x_3 + x_4 + x_5 = 4$.
Then $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 3, 0, 1)$ is a solution.

$+ + 111 + +1$ is a permutation of $\{4 \cdot 1, (5-1) \cdot +\}$

$(x_1, x_2, x_3, x_4, x_5) = (2, 1, 0, 1, 0)$ is a solution to $x_1 + x_2 + x_3 + x_4 + x_5 = 4$.

$11 + 1 + +1+$ is a permutation of $\{4 \cdot 1, (5-1) \cdot +\}$

A 4-combination of S is $\{a_1, a_1, a_2, a_4\}$

$+ + + + 4$ is a permutation of $\{4 \cdot 1, (5-1) \cdot +\}$

A 4-combination of S is $\{a_5, a_5, a_5, a_5\}$

$(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 0, 4)$ is a solution to $x_1 + x_2 + x_3 + x_4 + x_5 = 4$.