Math 150 Exam 1
October 4, 2006

Choose 7 from the following 9 problems. Circle your choices: 1 2 3 4 5 6 7 8 9
You may do more than 7 problems in which case your two unchosen problems can replace your lowest one or two problems at 2/3 the value as discussed in class.

1.) P(10, 7) = (10)(9)(8)(7)(6)(5)(4)

$C(10, 7) = \binom{10}{7} = \frac{10!}{7!3!} = \frac{(10)(9)(8)}{(3)(2)(1)} = (10)(3)(4) = 120$

The inversion sequence for the permutation 15243 is $0, 1, 2, 1, 0$

The permutation corresponding to the inversion sequence 3, 0, 2, 1, 0 is $2, 5, 4, 1, 3$

2.) $r(9, 2) =$ 9

$r(3, 3) =$ 6

Given that $\{x_{13}, x_{12}, x_7, x_1\}$ is a 4-combination of $\{x_{13}, x_{12}, ..., x_1, x_0\}$, Determine the combinations which come immediately before and after the combination $\{x_{13}, x_{12}, x_7, x_1\}$, using the base 2 generating scheme.

Before $\{x_{13}, x_{12}, x_7, x_1\}$: $\{x_{13}, x_{12}, x_7, x_0\}$:

1 1 0 0 0 0 1 0 0 0 0 0 1 0 - 1 = 1 1 0 0 0 0 1 0 0 0 0 0 0 1

After $\{x_{13}, x_{12}, x_7, x_1\}$: $\{x_{13}, x_{12}, x_7, x_1, x_0\}$

1 1 0 0 0 0 1 0 0 0 0 0 1 0 + 1 = 1 1 0 0 0 0 1 0 0 0 0 0 1 1

Determine the 4-combinations of $\{1, 2, ..., 14\}$ which come immediately before and after the the 4-combination $\{2, 8, 13, 14\}$ in lexicographical ordering.

Before $\{2, 8, 13, 14\}$: $\{2, 8, 12, 14\}$

After $\{2, 8, 13, 14\}$:$\{2, 9, 10, 11\}$

3.) In how many ways can 9 indistinguishable rooks be places on a 20-by-20 chessboard so that no rook can attack another rook?

$\frac{20!}{9!(11)!} \frac{20!}{11!}$

In how many ways can 9 rooks be places on a 20-by-20 chessboard so that no rook can attack another rook if no two rooks have the same color?

$\frac{20!}{9!(11)!}\frac{20!}{11!}9!$

4.) How many different circular permutations can be made using using 30 beads if you have 20 green beads, 9 blue beads and 1 red beads?

$\frac{29!}{20!\ 9!}$

5.) How many sets of 3 numbers each can be formed from the numbers $\{1, 2, 3, ..., 50\}$ if no two consecutive numbers are to be in a set?

Suppose we think of the 50 numbers as 50 sticks. The number of ways of removing 3 sticks such that no two are consecutive is the same as the number of integral solutions to $x_1 + x_2 + x_3 + x_4 = 47$ where $x_1, x_4 \geq 0$ and $x_2, x_3 \geq 1$. This is the same as the number of solutions to $x_1 + y_2 + 1 + y_3 + 1 + x_4 = 47$ where $x_1, x_4 \geq 0$, $y_2 = x_2 - 1 \geq 1 - 1 = 0$, $y_3 = x_3 - 1 \geq 1 - 1 = 0$. This is the same as the number of solutions to $x_1 + y_2 + y_3 + x_4 = 45$ where $x_1, x_4, y_2, y_3 \geq 0$.

Hence by thm 3.5.1, the answer is $\binom{45+4-1}{45} = \binom{48}{45} = \frac{48(47)(46)}{6}$

6.) Use the pigeonhole principle to prove that in a group of n people where $n > 1$, there are at least 2 people who have the same number of acquaintances. State where you use the pigeonhole principle.

Number the people 1 through n. We will assume that all acquaintances are mutual. We will also assume that you can't be your own acquaintance. Thus if person i has k_i acquaintances among the group of n people, $k_i \in \{0, ..., n-1\}$.

Case 1: There exists someone who knows everyone else. Then $k_i \in \{1, ..., n-1\}$ for $i = 1, ..., n$. Thus by the pigeonhole principle, there exists $i \neq j$ such that $k_i = k_j$.

Case 2: There does not exist someone who knows everyone else. Then $k_i \in \{0, ..., n-2\}$ for $i = 1, ..., n$. Thus by the pigeonhole principle, there exists $i \neq j$ such that $k_i = k_j$.

7.) Suppose $x, y \in \mathcal{Z}$. Define a relation on $\mathcal{Z}$ such that $x \sim y$ iff there exists $k \in \mathcal{Z}$ such that $x - y = 5k$. Show $\sim$ is an equivalence relation on $\mathcal{Z}$. What are the equivalence classes?

Claim: $\sim$ is reflexive.

$x - x = 5(0)$ and $0 \in \mathcal{Z}$. Thus $x \sim x$.

Claim: $\sim$ is symmetric.

Suppose $x \sim y$. Then there exists $k \in \mathcal{Z}$ such that $x - y = 5k$. Thus $y - x = 5(-k)$. $k \in \mathcal{Z}$ implies $-k \in \mathcal{Z}$. Thus $y \sim x$.

Claim: $\sim$ is transitive. Suppose $x \sim y$ and $y \sim z$. Then there exists $k \in \mathcal{Z}$ such that $x - y = 5k$. Also, there exists $n \in \mathcal{Z}$ such that $y - z = 5n$. Thus $x - z = x - y + y - z = 5k + 5n = 5(k + n)$. $k, n \in \mathcal{Z}$ implies $k + n \in \mathcal{Z}$. Thus $x \sim z$.

The equivalence classes are

$[0] = \{... - 10, -5, 0, 5, 10, ...\}$

$[1] = \{... - 9, -4, 1, 6, 11, ...\}$

$[2] = \{... - 8, -3, 2, 7, 12, ...\}$

$[3] = \{... - 7, -2, 3, 8, 13, ...\}$

$[4] = \{... - 6, -1, 4, 9, 14, ...\}$

8.) Let $X = \{1, 2, 3\}$. Define a partial order on $X \times X$ by $(x_1, y_1) \leq_x (x_2, y_2)$ iff $x_1 \leq x_2$ (for example $(1, 3) \leq_x (2, 1)$). Is $\leq_x$ reflexive? Is $\leq_x$ symmetric? Is $\leq_x$ antisymmetric? Is $\leq_x$ transitive? Is $\leq_x$ a partial order? Is $\leq_x$ an equivalence relation? Give a proof for each answer.

Claim: $\leq$ is reflexive.

Take $(x, y) \in X$. $x \leq x$. Thus $(x, y) \leq_x (x, y)$

Claim: $\leq_x$ NOT symmetric?

$(1, 2) \leq_x (2, 1)$ since $1 \leq 2$, but $(2, 1) \leq_x (1, 2)$ since $2 \not\leq 1$.

Claim: $\leq_x$ is NOT antisymmetric?

$(1, 2) \leq_x (1, 3)$ since $1 \leq 1$. $(1, 3) \leq_x (1, 2)$ since $1 \leq 1$. But $(1, 2) \neq (1, 3)$

Claim: $\leq$ is transitive.

Suppose $(x_1, y_1) \leq_x (x_2, y_2)$ and $(x_2, y_2) \leq_x (x_3, y_3)$.

$(x_1, y_1) \leq_x (x_2, y_2)$ implies $x_1 \leq x_2$. $(x_2, y_2) \leq_x (x_3, y_3)$ implies $x_2 \leq x_3$.

$x_1 \leq x_2$ and $x_2 \leq x_3$ implies $x_1 \leq x_3$. Thus $(x_1, y_1) \leq_x (x_3, y_3)$.

$\leq_x$ is NOT a partial order since it is not anti-symmetric.

$\leq_x$ is NOT an equivalence relation since it is not symmetric.

9.) Use a combinatorial argument to prove $\Sigma_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$

$\binom{2n}{n}$ = the number of ways to choose n elements from $\{1, ..., 2n\}$.

$\binom{n}{k}$ = the number of ways to choose k elements from $\{1, ..., n\}$.

$\binom{n}{n-k}$ = the number of ways to choose $n-k$ elements from $\{n+1, ..., 2n\}$.

Suppose A is an n-element subset of $\{1, ..., 2n\}$. Let $k = |A \cap \{1, ..., n\}|$.

Thus to choose an n-element subset of $\{1, ..., 2n\}$, we can first fix k and choose k elements from $\{1, ..., n\}$ and $n-k$ elements from $\{n+1, ..., 2n\}$. For a fixed k, the number of ways of choosing k elements from $\{1, ..., n\}$ and $n-k$ elements from $\{n+1, ..., 2n\}$ is $\binom{n}{k}\binom{n}{n-k}$. To get all n element subset of $\{1, ..., 2n\}$, we must do this for $k = 0, ..., n$. Thus the number of ways to choose n elements from $\{1, ..., 2n\} = \Sigma_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$.