Math 150 Final Exam
December 13, 2006

Choose 7 from the following 10 problems. Circle your choices: 1 2 3 4 5 6 7 8 9 10
You may do more than 7 problems in which case your unchosen problems can replace your lowest one or two problems at 2/3 the value as discussed in class.

1.) Show that every sequence $a_1, a_2, ..., a_{n^2+1}$ contains either an increasing or decreasing subsequence of length $n+1$.

See application 9 in section 3.2 (p. 76) for full proof.

2.) The Ramsey number $r(3,3) = 6$______________. Prove your answer.

See proof in book on p. 78.

Note you need to prove that if the edges of K_6 are colored red or blue, then there exists either a red triangle or a blue triangle.

You also need to prove that $r(3,3) > 5$. You can do this by giving an example of a coloring of K_5 which contains neither a red triangle nor a blue triangle. See Fig 3.2.

3.) Is the intersection $R \cap S$ of two equivalence relations R and S on a set X always an equivalence relation on X? Is the union $R \cup S$ of two equivalence relations R and S on a set X always an equivalence relation on X? Prove your answer.

See HW problem ch4: 49

4.) Find the number of integral solutions to the equation $x_1 + x_2 + x_3 + x_4 = 60$ such that $0 \leq x_1 \leq 10$, $1 \leq x_2 \leq 5$, $x_3 \geq -2$, and $x_4 \geq 4$.

Let $y_1 = x_1$. Let $y_2 = x_2 - 1$. Let $y_3 = x_3 + 2$. Let $y_4 = x_4 - 4$.

$x_1 + x_2 - 1 + x_3 + 2 + x_4 - 4 = 60 - 1 + 2 - 4$

The number of integral solutions to the equation $x_1 + x_2 + x_3 + x_4 = 60$ such that
$0 \leq x_1 \leq 10$, $1 \leq x_2 \leq 5$, $x_3 \geq -2$, and $x_4 \geq 4$ is the same as

the number of integral solutions to the equation $y_1 + y_2 + y_3 + y_4 = 57$ such that
$0 \leq y_1 \leq 10$, $0 \leq y_2 \leq 4$, $y_3 \geq 0$, and $y_4 \geq 0$.

Method 1: generating function (note we did not need to change lower bounds to use generating functions, but since we are not use to generating functions with a couple of negative exponents, I changed the lower bound. A generating function would have worked just as well without changing the lower bounds.

$(x^0+x^1+...+x^{10})(x^0+x^1+...+x^4)(x^0+x^1+...)(x^0+x^1+...) = (\frac{x^{11}-1}{x-1})(\frac{x^5-1}{x-1})(\frac{1}{1-x})(\frac{1}{1-x}) = \frac{(x^{11}-1)(x^5-1)}{(x-1)^4}) = (x^{16}-x^{11}-x^5+1)\Sigma_{k=0}^{\infty}\begin{pmatrix}4+k-1\\k\end{pmatrix}x^k$

We need $y_1+y_2+y_3+y_4=57$, thus we need the coefficient of x^{57}.

$[\begin{pmatrix}44\\41\end{pmatrix}-\begin{pmatrix}49\\46\end{pmatrix}-\begin{pmatrix}55\\52\end{pmatrix}+\begin{pmatrix}60\\57\end{pmatrix}]x^{57}$

$=(45)(44)(43)-(49)(47)(46)-(56)(55)(54)+(60)(59)(58)$

Method 2: Inclusion-Exclusion

Let S = solutions to $y_1+y_2+y_3+y_4=57$ such that $0 \leq y_1$, $0 \leq y_2$, $y_3 \geq 0$, and $y_4 \geq 0$.

$|S| = \begin{pmatrix}57+4-1\\57\end{pmatrix} = \begin{pmatrix}60\\57\end{pmatrix}$

Let A_1 = solutions to $y_1+y_2+y_3+y_4=57$ such that $11 \leq y_1$, $0 \leq y_2$, $y_3 \geq 0$, and $y_4 \geq 0$.

$|A_1|$ = the number of solutions to $y_1+y_2+y_3+y_4=46$ such that $0 \leq y_1$, $0 \leq y_2$, $y_3 \geq 0$, and $y_4 \geq 0$

Thus $|A_1| = \begin{pmatrix}46+4-1\\46\end{pmatrix} = \begin{pmatrix}49\\46\end{pmatrix}$

Let A_2 = solutions to $y_1+y_2+y_3+y_4=57$ such that $0 \leq y_1$, $5 \leq y_2$, $y_3 \geq 0$, and $y_4 \geq 0$.

$|A_2|$ = the number of solutions to $y_1+y_2+y_3+y_4=52$ such that $0 \leq y_1$, $0 \leq y_2$, $y_3 \geq 0$, and $y_4 \geq 0$

Thus $|A_2| = \begin{pmatrix}52+4-1\\52\end{pmatrix} = \begin{pmatrix}55\\52\end{pmatrix}$

$|A_1 \cap A_2|$ = the number of solutions to $y_1+y_2+y_3+y_4=41$ such that $0 \leq y_1$, $0 \leq y_2$, $y_3 \geq 0$, and $y_4 \geq 0$

Thus $|A_2| = \begin{pmatrix}41+4-1\\41\end{pmatrix} = \begin{pmatrix}44\\41\end{pmatrix}$

Thus $\begin{pmatrix}60\\57\end{pmatrix}-\begin{pmatrix}49\\46\end{pmatrix}-\begin{pmatrix}55\\52\end{pmatrix}+\begin{pmatrix}44\\41\end{pmatrix}$

5.) Let D_n be the number of derangements of $\{1,2,...,n\}$. Determine a formula for D_n. Prove your answer.

See Thm 6.3.1

6.) Solve the recurrence relation $h_n = 2h_{n-1} + 3^n$ with initial value $h_0 = 4$

A.) Solve homogeneous recurrence relation: $h_n = 2h_{n-1}$.

Guess $h_n = q^n$. Then $h_n = 2h_{n-1}$ implies $q^n = 2q^{n-1}$. Hence $q = 2$.

Thus the general solution to the homogeneous recurrence relation is $c(q^n) = c(2^n)$.

B.) Guess a solution to the non-homogeneous recurrence relation $h_n = 2h_{n-1} + 3^n$. Note we only need one solution for the non-homogeneous recurrence relation.

Try a multiple of 3^n. Suppose $h_n = a(3^n)$.

Then $h_n = 2h_{n-1} + 3^n$ implies $a(3^n) = 2a(3^{n-1}) + 3^n$

Thus $3(a-1)(3^{n-1}) = 2a(3^{n-1})$

Hence $3a - 3 = 2a$.

$a = 3$. Thus a solution to the non-homogeneous recurrence relation is $h_n = 3(3^n) = 3^{n+1}$

HENCE, the general solution to the non-homogeneous recurrence relation is
$h_n = c(2^n) + 3^{n+1}$

C.) Use initial conditions to find c:

$h_0 = 4$ implies $4 = c(2^0) + 3^1$. Hence $c = 1$.

Thus $h_n = 2^n + 3^{n+1}$

Check: $h_0 = 2^0 + 3^1 = 4$.

$h_n - 2h_{n-1} - 3^n = 2^n + 3^{n+1} - 2[2^{n-1} + 3^n] - 3^n = 2^n + 3^{n+1} - 2^n - 2(3^n) - 3^n = 3^{n+1} - 3(3^n) = 0$.

7a.) Determine the generating function for the number h_n of n-combinations of fruit consisting of apples, oranges, bananas, pears, and kiwis in which there are an odd number of apples, the number of oranges is a multiple of 4, the number of bananas is at most 3, the number of pears is 0 or 1, and there are at least 2 kiwis.

$(x + x^3 + x^5 + ...)(x^0 + x^4 + x^8 + ...)(x^0 + x^1 + x^2 + x^3)(x^0 + x^1)(x^2 + x^3 + x^4 + ...)$

$= (\frac{x}{1-x^2})(\frac{1}{1-x^4})(\frac{x^4-1}{x-1})(1+x)(\frac{x^2}{1-x})$

$= (\frac{-x^3}{(1-x)(1+x)})(\frac{1}{x-1})(1+x)(\frac{1}{1-x})$

$= \frac{x^3}{(1-x)^3} = x^3 \Sigma_{k=0}^{\infty} \begin{pmatrix} n+3-1 \\ n \end{pmatrix} x^n = \Sigma_{k=0}^{\infty} \begin{pmatrix} n+3-1 \\ n \end{pmatrix} x^{n+3}$

Thus coefficient of x^n is $\begin{pmatrix} n-1 \\ n-3 \end{pmatrix}$. Thus $h_n = \begin{pmatrix} n-1 \\ n-3 \end{pmatrix}$ is the number of n-combinations.

7b.) Find a formula for h_n.

$$h_n = \binom{n-1}{n-3}$$

8a.) Find the number of partitions of 6 distinguishable objects into 3 nonempty distinguishable boxes.

Use Inclusion-Exclusion.

Let S = the set of partitions of 6 distinguishable objects into 3 distinguishable.

Let A_i = the set of partitions of 6 distinguishable objects into 3 distinguishable boxes where box i is empty.

Note that for each object, there are 3 choices for which box to place the object. Thus $|S| = 3^6$

To determine $|A_1|$, not that there are 2 choices for which box to place the object as we can only place objects in boxes 2 and 3 as box 1 must remain empty. Thus $|A_1| = 2^6$

Similary $|A_i| = 2^6$ for $i = 1, 2, 3$

$|A_1 \cap A_2| = 1$ as all objects must be placed in box 3.

Similary $|A_i \cap A_j| = 1$ for $i, j = 1, 2, 3, i \neq j$

$|A_1 \cap A_2 \cap A_3| = 0$ as we can't place 6 objects in boxes and have all boxes empty.

By Inclusion-Exclusion, the number of partitions of 6 distinguishable objects into 3 nonempty distinguishable boxes is

$$|S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - |A_1 \cap A_2 \cap A_3|$$

$$= 3^6 - \Sigma 2^6 + \Sigma 1 - 0$$

$$= 3^6 - 3(2^6) + \binom{3}{2} - 0$$

$$= 3^6 - 3(2^6) + 3$$

8b.) Find the difference table for $h_n = n^2 + 1$: From Chapter 8, not covered this semester.

8c.) $\Sigma_{k=0}^{n} h_k =$ _______________: From Chapter 8, not covered this semester.

9a.) Find the number of subsets of $\{1, 2, 3, ..., 10\}$.

2^n

9b.) Find the number of subsets of $\{1, 2, 3, ..., 10\}$ which have exactly 8 elements .

$\binom{10}{8} = \frac{10!}{8!2!}$

9c.) Find the number of permutations of $\{1, 2, 3, ..., 10\}$ which have exactly 8 elements.

$P(10, 8) = \frac{10!}{2!}$

9d.) Find the number of permutations of $\{3 \cdot a, 4 \cdot b, 1 \cdot c\}$ which have exactly 8 elements.

$\frac{8!}{3!4!1!}$

Note that $3 + 4 + 1 = 8$. If we had fewer than 8 elements, the number of 8-permutations is 0. If we had more than 8 elements, this would be a harder problem which could be solved by (1) breaking into cases or (2) exponential generating function.

9e.) Find the number of partitions of 25 indistinguishable objects into 10 distinguishable boxes.

I.e, how many solutions are there to $x_1 + x_2 + ... + x_{10} = 25$ where x_i = the number of objects placed in box i.

I.e, the number of permutations of $\{25 \cdot 1, 9 \cdot +\}$

Thus answer is $\binom{25+9}{25} = \binom{34}{25}$

10a.) Expand $(x - 2y)^6$ using the binomial theorem.

$\binom{6}{0} x^6 + \binom{6}{1} x^5(-2y) + \binom{6}{2} x^4(-2y)^2 + \binom{6}{3} x^3(-2y)^3 + \binom{6}{4} x^2(-2y)^4 + \binom{6}{5} x(-2y)^5 + \binom{6}{6} (-2y)^6$

$= x^6 - 6x^5(2y) + 15x^4(2y)^2 - 20x^3(2y)^3 + 15x^2(2y)^4 - 6x(2y)^5 + (2y)^6$

10b.) What is the coefficient of $x^4y^3z^2$ in the expansion of $(x - y + 3z)^9$: $\underline{-(81)(140)}$

$\binom{9}{4\ 3\ 2} x^4(-y)^3(3z)^2 = -\frac{(9)(8)(7)(6)(5)}{3!2!}(9x^4y^3z^2) = -(81)(4)(7)(5)(x^4y^3z^2) = -(81)(140)(x^4y^3z^2)$

10c.) What is the coefficient of $x^3y^3z^2$ in the expansion of $(x - y + 3z)^9$: $\underline{0}$

Note $3 + 3 + 2 \neq 9$. Thus coefficient is 0.

10d.) The inversion sequence for the permutation 615423 is $\underline{133210}$

10e.) The permutation corresponding to the inversion sequence 5, 1, 3, 2, 1, 0 is $\underline{625431}$