Thm 2.2.1 Pigeonhole Principle (strong form): Let $q_1, q_2, ..., q_n$ be positive integers. If $q_1 + q_2 + ... + q_n - n + 1$ objects are put into n boxes, then for some i the ith box contains at least q_i objects

Cor: If $q_i = r$ for all i, then if $n(r-1)+1$ objects are put into n boxes, then there exists a box containing at least r objects.

Cor: If $\frac{m_1 + ..., m_n}{n} > r - 1$, then there exists an i such that $m_i \geq r$.

Appl 9: Show that every sequence $a_1, a_2, ..., a_{n^2+1}$ contains either an increasing or decreasing subsequence of length $n+1$.

Example ($n = 2$):

$a_1 = 8$, $a_2 = 4$, $a_3 = 10$, $a_4 = 6$, $a_5 = 4$

Need $n + 1$ objects in our subsequence. Suppose $r = n + 1$.

Hence might need $n(r - 1) + 1 = n(n + 1 - 1) + 1 = n^2 + 1$ objects in n boxes in order to obtain at least $r = n+1$ objects in one of the boxes.

Let m_k = length of largest increasing subsequence beginning with a_k.

8   8, 10 $m_1 = 2$

4   4, 10   4, 6   4, 4 $m_2 = 2$

10   $m_3 = 1$ 6 $m_4 = 1$ 4 $m_5 = 1$

Proof: Let m_k = length of largest increasing subsequence beginning with a_k, $k = 1, ..., n^2 + 1$.

Suppose there exists an $m_k \geq n + 1$. Then there exists an increasing subsequence of length $m_k \geq n + 1$. Hence there exists an increasing subsequence of length $n + 1$.

Suppose $m_k < n + 1$. Then $m_k = 1, 2, ...,$ or n.

Hence there exists an i such that $m_k = i$ for $n + 1$ a_k's.

There exists $a_{k_1}, a_{k_2}, ..., a_{k_{n+1}}$ such that

$$m_{k_1} = m_{k_2} = ... = m_{k_{n+1}} = i$$

Show $a_{k_1}, a_{k_2}, ..., a_{k_{n+1}}$ is a decreasing sequence.

Suppose not. Then there exists a j such that $a_{k_j} > a_{k_{j+1}}$.

$\exists$ an increasing subsequence of length i starting at a_{k_j}

There does not exist an increasing subsequence of length $i+1$ starting at a_{k_j}

$\exists$ an increasing subsequence of length i starting at $a_{k_{j+1}}$

There does not exist an increasing subsequence of length $i+1$ starting at $a_{k_{j+1}}$

Suppose $a_{k_{j+1}}, a_{h_2}, a_{h_3}, ..., a_{h_i}$ is an increasing subsequence of length i.

Then $a_{k_j}, a_{k_{j+1}}, a_{h_2}, a_{h_3}, ..., a_{h_i}$ is an increasing subsequence of length $i + 1$, a contradiction.