Section 4.3/4.4 Exponential growth/decay

Thm 8: Suppose c, k are constants. Then

$$\frac{dy}{dx} = ky \text{ if and only if } y = ce^{kx}$$

I.e., If the (instantaneous) rate of change of y with respect to x is proportional to y, then

Section 4.3: $k > 0$ implies exponential growth.

Section 4.4: $k < 0$ implies exponential decay.

For simplicity, take $k > 0$. Then in section 4.4,

Section 4.4 version of Thm 8: Suppose c, k are constants. Then

$$\frac{dy}{dx} = -ky \text{ if and only if } y = ce^{-kx}$$

I.e., If the (instantaneous) rate of change of y with respect to x is proportional to y, then we have exponential decay since $-k < 0$.

Initial Value Problem (IVP): $\frac{dy}{dx} = ky$, $y(x_0) = y_0$

$$\text{if and only if } y = ce^{kx},\ y_0 = ce^{kx_0} \Rightarrow c = \frac{y_0}{e^{kx_0}}$$

$$\text{Hence } y = ce^{kx} \text{ where } c = \frac{y_0}{e^{kx_0}}$$

Precalculus:

Let $P(0) = P_0$

Section 4.3, $k > 0$:
Doubling time = generation time: If $P(t) = P_0 e^{kt}$, then at what time t is $P(t) = 2P_0$

$2P_0 = P_0 e^{kt} \Rightarrow 2 = e^{kt} \Rightarrow ln(2) = ln(e^{kt}) = kt \Rightarrow t = \frac{ln(2)}{k}$

Section 4.4, $-k < 0$:
Half life: If $P(t) = P_0 e^{-kt}$, then at what time t is $P(t) = \frac{1}{2} P_0$

$$\frac{1}{2} P_0 = P_0 e^{-kt} \Rightarrow \frac{1}{2} = e^{-kt}$$
$$\Rightarrow 2 = e^{kt} \Rightarrow ln(2) = ln(e^{kt}) = kt \Rightarrow t = \frac{ln(2)}{k}$$

You do NOT need to know the following for either exam 1b or exam 2:

Thm 11: Newton's law of cooling

The rate of change of temperature T with respect to time t is given by

$$\frac{dT}{dt} = -k(T - S)$$

where $k > 0$ is the proportionality constant, and S is the constant temperature of the surrounding medium.

Hence $T(t) = P_0 e^{-kt} + S$ where $P_0 = T(0) - C$

Proof: $T' = -k(T - S) \Rightarrow \frac{T'}{T-S} = -k \Rightarrow ln|T - S| = -kt + C_1$, for some constant C_1

$|T - S| = e^{ln|T-S|} = e^{-kt+C_1} = e^{-kt} e^{C_1}$

Hence $T - S = C_2 e^{-kt}$ for some constant C_2

When $t = 0$, $T(0) - S = C_2 e^0 = C_2$

Hence $T(t) = P_0 e^{-kt} + S$ where $P_0 = T(0) - C$