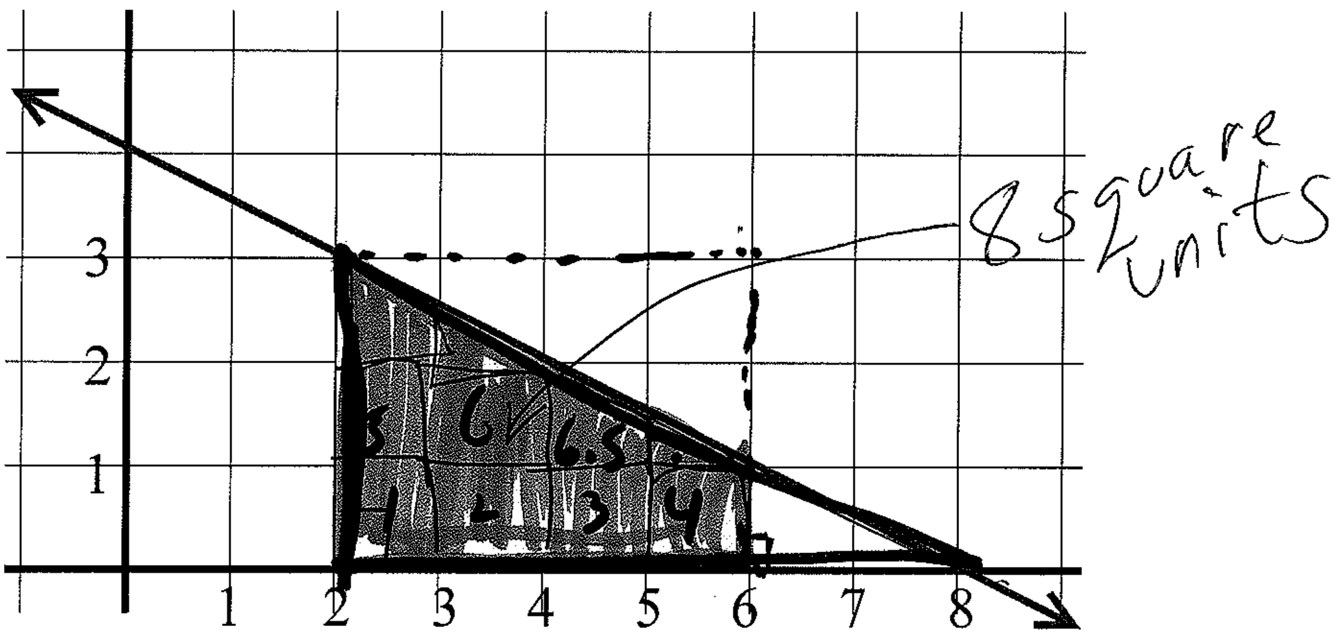
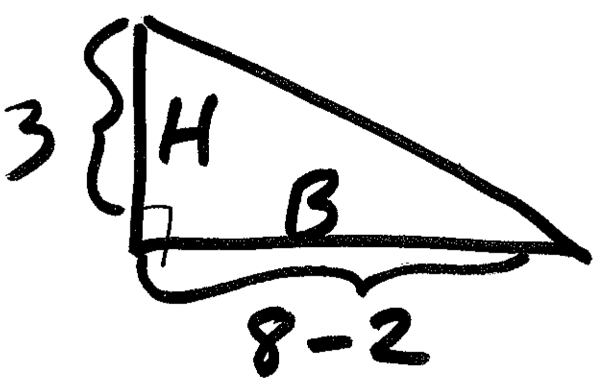
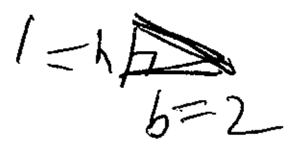
Section 5.2

Find the area under the curve $f(x) = -\frac{1}{2}x + 4$, above the $x-$axis and between $x = 2$ and $x = 6$.



Method 1: In this case our function is very simple, so we can determine the area without calculus:





$$\frac{1}{2}BH - \frac{1}{2}bh = \frac{1}{2}(8-2)(3) - \frac{1}{2}(2)(1)$$

$$= \frac{1}{\cancel{2}}\overset{3}{\cancel{6}}\cdot 3 - \frac{1}{\cancel{2}}\cancel{2}\cdot 1 = 9 - 1 = 8$$

under-estimate

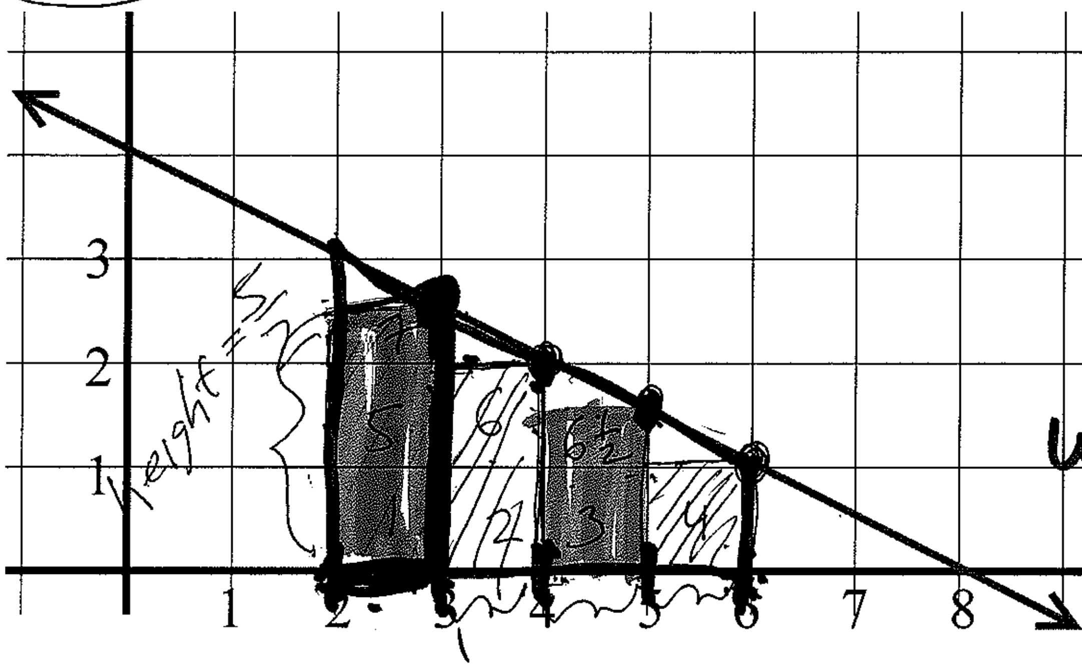
Method 2: Estimate using rectangles.

Inscribed rectangles with $\Delta x = 1$:

$6-2 = 4$

$\frac{4}{4}$ ← length $= 6-2$; ← # of rectangle

width $= \frac{4}{4} = 1$



width = 1

$$\sum f(x_i)\Delta x = \sum_{i=3}^{6} f(i)(1) = \sum_{i=1}^{4} f(i+2)(1)$$

$f(3)(1) + f(4)(1) + f(5)(1) + f(6)(1) =$ width

height width height

$= [-\frac{1}{2}(3)+4](1) + [-\frac{1}{2}(4)+4](1)$

$+[-\frac{1}{2}(5)+4](1) + [-\frac{1}{2}(6)+4](1)$

$= \boxed{\frac{5}{2}(1) + 2(1) + \frac{3}{2}(1) + 1(1) = 7} < 8$

since inscribed

width width

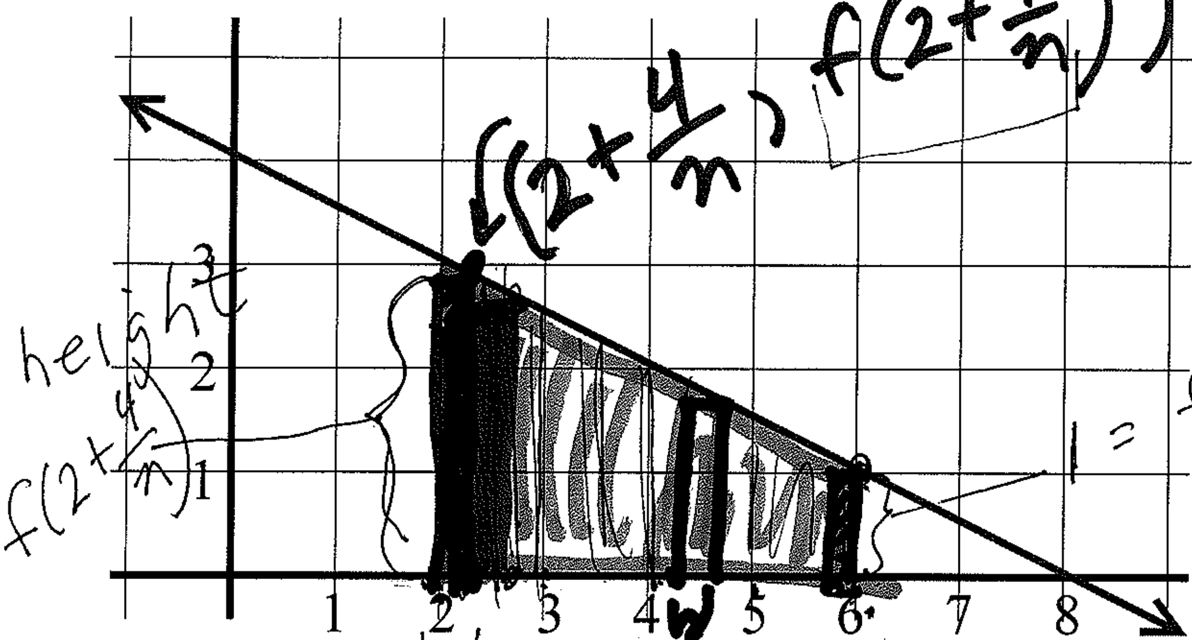
height height

change in x = width

Inscribed rectangles with $\Delta x = \frac{6-2}{n} = \frac{4}{n}$:

If have n rectangles

$\frac{6-2}{n} = \frac{4}{n}$



width $\frac{4}{n}$

width $= \frac{4}{n}$.

height · width $\qquad$ height · width $\qquad$ height width

$f\left(2+\frac{4}{n}\right) \cdot \frac{4}{n} + \left[f\left(2+\frac{4}{n}+\frac{4}{n}\right)\right]\left(\frac{4}{n}\right) + \ldots\ldots f(6)\left(\frac{4}{n}\right)$

Sum of the area of rectangles

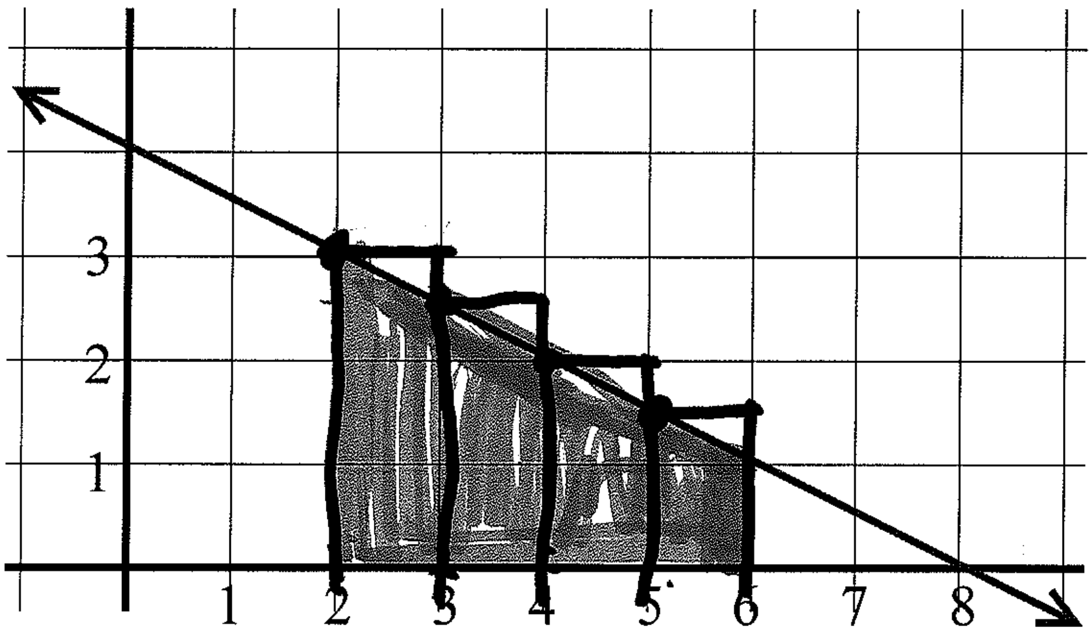
= sum of heights × widths $= \sum f(x_i)\left(\frac{4}{n}\right)$

Total area estimate $= \sum f(x_i) \cdot \Delta x$

$\qquad$ height $\qquad$ · width

over-estimate

Circumscribed rectangles with $\Delta x = 1$:



$$\sum_{i=1}^{4} f(x_i)(1) = \sum_{i=1}^{4} f(i+1)(1)$$

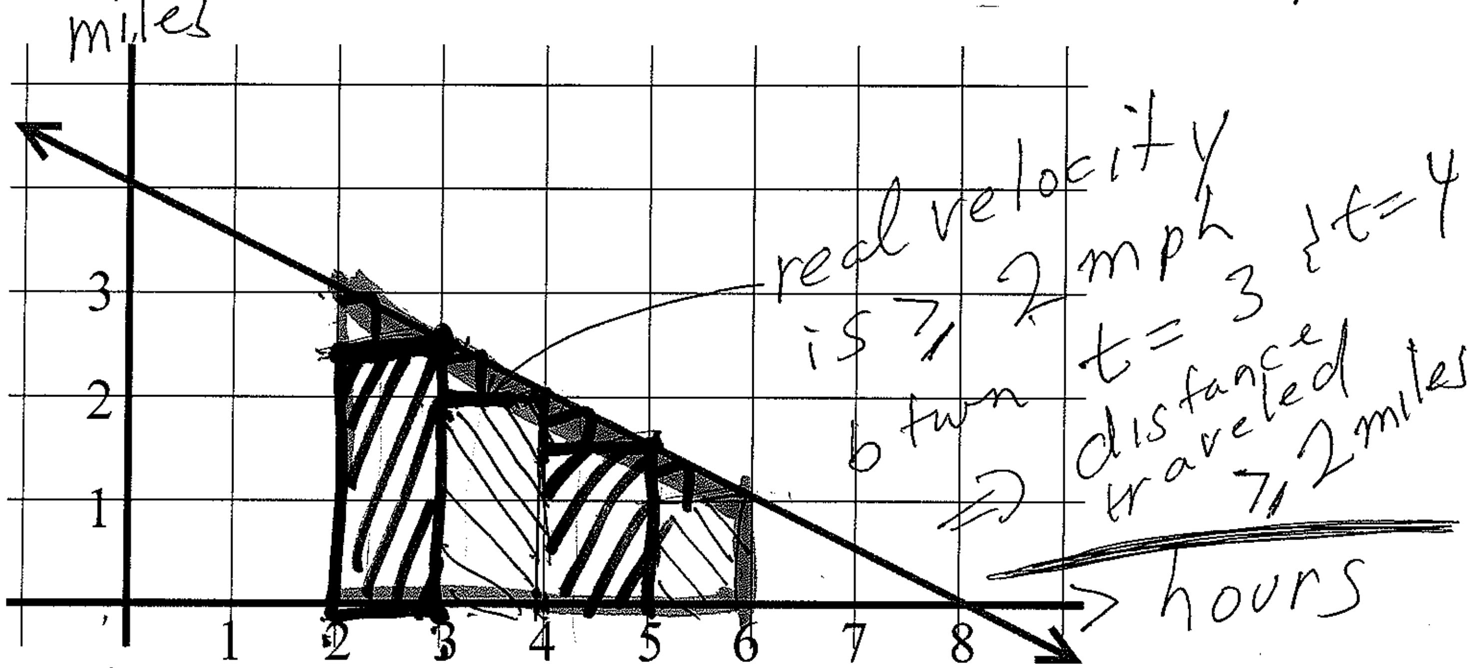
$f(2)(1) + f(3)(1) + f(4)(1) + f(5)(1) =$

$$= [-\tfrac{1}{2}(2) + 4](1) + [-\tfrac{1}{2}(3) + 4](1) + [-\tfrac{1}{2}(4) + 4](1) + [-\tfrac{1}{2}(5) + 4](1)$$

$$= 3 + \tfrac{5}{2}(1) + 2(1) + \tfrac{3}{2}(1) = 9 > 8$$

Estimate the distance traveled between $t = 2$ and $t = 6$ if the velocity is given by the function $f(t) = -\frac{1}{2}t + 4. = v(t)$ mph

Estimate using inscribed rectangles with $\Delta t = 1$:



Under estimate of distance traveled = sum of areas of rectangles

$f(3)(1) + f(4)(1) + f(5)(1) + f(6)(1) =$

$= [-\frac{1}{2}(3) + 4](1) + [-\frac{1}{2}(4) + 4](1)$
$\quad + [-\frac{1}{2}(5) + 4](1) + [-\frac{1}{2}(6) + 4](1)$

$= \frac{5}{2}(1) + 2(1) + \frac{3}{2}(1) + 1(1) = 7$ miles

v t height width

velocity × time = distance

$f>0$

Defn: $\boxed{\int_a^b f(x)dx = lim_{n\to\infty}\Sigma_{i=1}^n \underbrace{\overbrace{f(x_i)}^{\text{height}}}\ \underbrace{\overbrace{\Delta x}^{\text{width}}}}$ = Area under curve

If f is continuous, can use inscribed rectangles, circumscribed rectangles, all left-hand endpoints, all right-hand endpoints, or all midpoints, etc.

If $\Delta x = \frac{b-a}{n}$ and if right-hand endpoints are used, then $x_i = a + i\Delta x = a + \frac{(b-a)i}{n}$

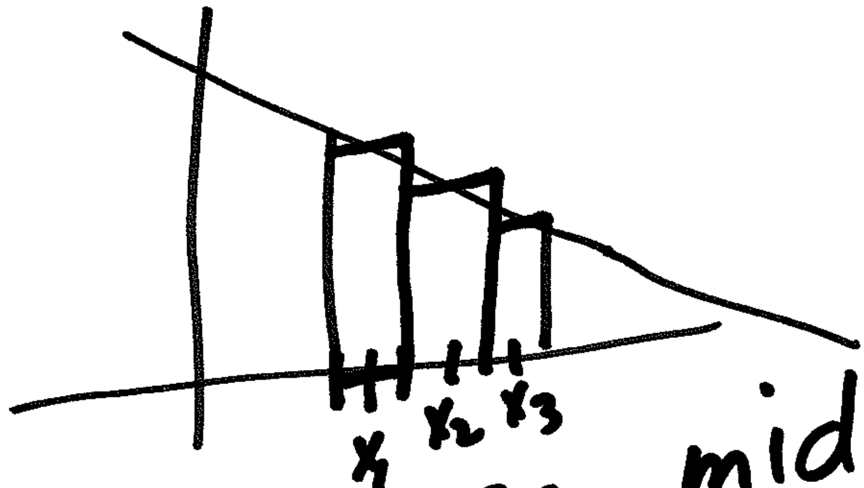
$$\int_a^b f(x)dx = lim_{n\to\infty}\Sigma_{i=1}^n \overbrace{f\left(a + \frac{(b-a)i}{n}\right)}^{\text{height}}\overbrace{\left(\frac{b-a}{n}\right)}^{\text{width}}$$

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n} \underbrace{\overbrace{f(x_i)}^{\text{height}}\,\overbrace{\Delta x}^{\text{width}}}_{\text{area of rectangle}}\right] = \int_a^b f(x)dx$$

Add areas of all rectangles

estimated area using n rectangles

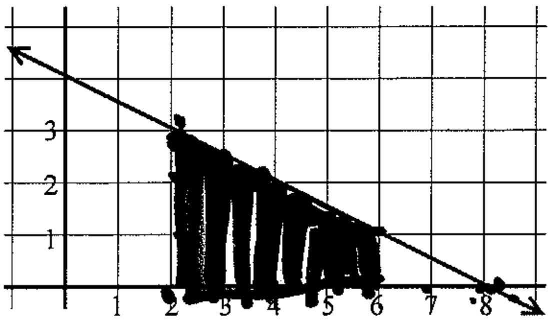
limit equals to actual area if $f>0$



can use midpts instead of ~~circut~~ inscribed or circumscribed rectangles

can use all right-hand endpts

Find the distance traveled between $t = 2$ and $t = 6$ if the velocity is given by the function $f(t) = -\frac{1}{2}t + 4$.



$= \text{area}$

Method 1: In this case our function is very simple, so we can determine the area without calculus:

$\frac{1}{2}BH - \frac{1}{2}bh$

$\frac{1}{2}(8-2)(3) - \frac{1}{2}(2)(1) = 9 - 1 = 8$

Method 2: Use calculus by estimating with rectangles and taking limit.

$$\begin{aligned}
\text{Area} = &= lim_{n\to\infty}\Sigma_{i=1}^{n} f\left(a + \tfrac{(b-a)i}{n}\right)\left(\tfrac{b-a}{n}\right) \\
&= lim_{n\to\infty}\Sigma_{i=1}^{n} f\left(2 + \tfrac{4i}{n}\right)\left(\tfrac{4}{n}\right) \\
&= lim_{n\to\infty}\Sigma_{i=1}^{n} \left[-\tfrac{1}{2}\left(2 + \tfrac{4i}{n}\right) + 4\right]\left(\tfrac{4}{n}\right) = 8
\end{aligned}$$

Method 3 (section 5.3): Use calculus by integrating.

$$\begin{aligned}
&\int_2^6 \left(-\tfrac{1}{2}t + 4\right)dt = \left(-\tfrac{1}{4}t^2 + 4t\right)\Big|_2^6 \\
&= \left(-\tfrac{1}{4}(6)^2 + 4(6)\right) - \left(-\tfrac{1}{4}(2)^2 + 4(2)\right) \\
&= -9 + 24 - (-1 + 8) = 15 - 7 = 8 \checkmark
\end{aligned}$$

Example:

$\int_2^6 (-\frac{1}{2}t + 4)dt = = lim_{n\to\infty} \Sigma_{i=1}^n f(t_i)\Delta t$

$\Delta t = \frac{6-2}{n} = \frac{4}{n}$ (using n equal subintervals)

$t_i = 2 + i\Delta t = 2 + \frac{4i}{n}$ (using right-hand endpoints)

$\int_2^6 (-\frac{1}{2}t + 4)dt = lim_{n\to\infty} \Sigma_{i=1}^n f(2 + \frac{4i}{n})(\frac{4}{n})$

$= lim_{n\to\infty} \Sigma_{i=1}^n [-\frac{1}{2}(2 + \frac{4i}{n}) + 4](\frac{4}{n})$

$= lim_{n\to\infty} \Sigma_{i=1}^n [-1 - \frac{2i}{n} + 4](\frac{4}{n})$

$= lim_{n\to\infty} \Sigma_{i=1}^n [3 - \frac{2i}{n}](\frac{4}{n})$

$= lim_{n\to\infty} \Sigma_{i=1}^n [\frac{12}{n} - \frac{8i}{n^2}]$

$= lim_{n\to\infty} (\Sigma_{i=1}^n \frac{12}{n} - \Sigma_{i=1}^n \frac{8i}{n^2})$

$= lim_{n\to\infty} (12 - \frac{8}{n^2} \Sigma_{i=1}^n i)$

$= lim_{n\to\infty} (12 - \frac{8}{n^2} \frac{n(n+1)}{2})$

$= lim_{n\to\infty} (12 - \frac{4n^2+4n}{n^2})$

$= lim_{n\to\infty} (12 - 4 - \frac{4}{n}) = 8$

FYI $\sum_{i=1}^{100} i = \frac{10100}{2}$

$1 + 2 + 3 + \ldots\ldots + 98 + 99 + 100$

$100 + 99 + 98 + \qquad + 3 + 2 + 1$

$101 + 101 + 101 + \ldots\ldots + 101 + 101 + 101$

$= (100)(101) = 10100$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1 $\qquad\qquad\qquad\qquad$ n

n $\qquad\qquad\qquad\qquad$ 1