

Let  $p : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ ,  $p(r, \theta) = (r\cos\theta, r\sin\theta)$ .

$$\text{Then } Dp(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ ,  $f(x, y) = x^2 + y^2 + 3y$

$$\begin{aligned} (f \circ p)(r, \theta) &= f(p(r, \theta)) = f(r\cos\theta, r\sin\theta) \\ &= r^2\cos^2\theta + r^2\sin^2\theta + 3r\sin\theta = r^2 + 3r\sin\theta \end{aligned}$$

**Note we used the function  $p$  to convert  $x^2 + y^2 + 3y$  into polar coordinates.**

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Use the chain rule to calculate  $D(f \circ p)(r, \theta)$ :

$$f(x, y) = x^2 + y^2 + 3y. \text{ Thus, } Df(x, y) = (2x \quad 2y + 3)$$

$$\begin{aligned} D(f \circ p)(r, \theta) &= (Df)(p(r, \theta))Dp(r, \theta) \\ &= (2r\cos\theta \quad 2r\sin\theta + 3) \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \\ &= (2r\cos^2\theta + 2r\sin^2\theta + 3\sin\theta \quad -2r^2\sin\theta\cos\theta + 2r^2\sin\theta\cos\theta + 3r\cos\theta) \\ &= (2r + 3\sin\theta \quad 3r\cos\theta) \end{aligned}$$

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$$\begin{aligned} D(f \circ p)(r, \theta) &= (Df)(p(r, \theta))Dp(r, \theta) \\ &= \left( \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= \left( \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \quad -\frac{\partial f}{\partial x}r\sin\theta + \frac{\partial f}{\partial y}r\cos\theta \right) \end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial r} &= \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= -r \sin\theta \frac{\partial}{\partial x} + r \cos\theta \frac{\partial}{\partial y}\end{aligned}$$

$$\begin{pmatrix} \cos\theta & \sin\theta & \frac{\partial}{\partial r} \\ -r \sin\theta & r \cos\theta & \frac{\partial}{\partial \theta} \end{pmatrix} \rightarrow \begin{pmatrix} r \cos\theta \sin\theta & r \sin^2\theta & r \sin\theta \frac{\partial}{\partial r} \\ -r \cos\theta \sin\theta & r \cos^2\theta & \cos\theta \frac{\partial}{\partial \theta} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} r \cos\theta \sin\theta & r \sin^2\theta & r \sin\theta \frac{\partial}{\partial r} \\ 0 & r & \cos\theta \frac{\partial}{\partial \theta} + r \sin\theta \frac{\partial}{\partial r} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \cos\theta & \sin\theta & \frac{\partial}{\partial r} \\ 0 & 1 & \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} + \sin\theta \frac{\partial}{\partial r} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \cos\theta & 0 & \frac{\partial}{\partial r} - \sin\theta \left[ \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} + \sin\theta \frac{\partial}{\partial r} \right] \\ 0 & 1 & \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} + \sin\theta \frac{\partial}{\partial r} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \cos\theta & 0 & \frac{\partial}{\partial r} - \sin\theta \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} - \sin^2\theta \frac{\partial}{\partial r} \\ 0 & 1 & \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} + \sin\theta \frac{\partial}{\partial r} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \cos\theta & 0 & -\sin\theta \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} + (1 - \sin^2\theta) \frac{\partial}{\partial r} \\ 0 & 1 & \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} + \sin\theta \frac{\partial}{\partial r} \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} + \cos^2\theta \frac{\partial}{\partial r} \\ 0 & 1 & \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} + \sin\theta \frac{\partial}{\partial r} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \frac{-\sin\theta}{r} \frac{\partial}{\partial \theta} + \cos\theta \frac{\partial}{\partial r} \\ 0 & 1 & \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} + \sin\theta \frac{\partial}{\partial r} \end{pmatrix} \blacksquare$$

$$\begin{aligned}\frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}\end{aligned}$$