3.4 $\frac{\partial}{\partial x} : \{f : \mathbb{R}^n \to \mathbb{R} \mid f \text{ diff}\} \to \{g : \mathbb{R}^n \to \mathbb{R}\}$

Ex $f(x, y, z) = x^2 y z$

$$\frac{\partial f}{\partial x}(x, y, z) = 2xyz$$

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad \frac{\partial f}{\partial x} : \mathbb{R}^3 \to \mathbb{R}$$

$$\underbrace{\frac{\partial}{\partial x}}(f) \to \frac{\partial f}{\partial x}$$

$$(x^2, x^3, x^4) = \vec{i}\,x^2 + \vec{j}\,x^3 + \vec{k}\,x^4$$

is a vector of fns

Can also view this vector as a fn

$$f(x) = (x^2, x^3, x^4)$$
$$f : \mathbb{R} \to \mathbb{R}^3$$

Del operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$= \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + k\,\frac{\partial}{\partial z}$$

is a vector of functions

$$\nabla : \{f : \mathbb{R}^n \to \mathbb{R} \mid f \text{ diff}\} \to \{g : \mathbb{R}^n \to \mathbb{R}^n\}$$

$$\nabla f = \nabla f$$

Input : diff $f : \mathbb{R}^3 \to \mathbb{R}$

Output : $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) : \mathbb{R}^3 \to \mathbb{R}^3$

Ex: Input : $f : \mathbb{R}^3 \to \mathbb{R}$

$f(x y z) = x^2 y z$

Output : $\nabla f = (2xyz,\ x^2 z,\ x^2 y)$

$$\nabla f : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\nabla : \{f : \mathbb{R}^n \to \mathbb{R}\} \to \{\nabla f : \mathbb{R}^n \to \mathbb{R}^n\}$$

Input: scalar field

Output: vector field

Divergence

$$F : X \subset \mathbb{R}^n \to \mathbb{R}^n \quad \text{vector field}$$

$$F = (F_1, \ldots, F_n)$$

divergence of $F = \nabla \cdot F$

$$\left(\frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_n}\right) \cdot (F_1, \ldots, F_n)$$

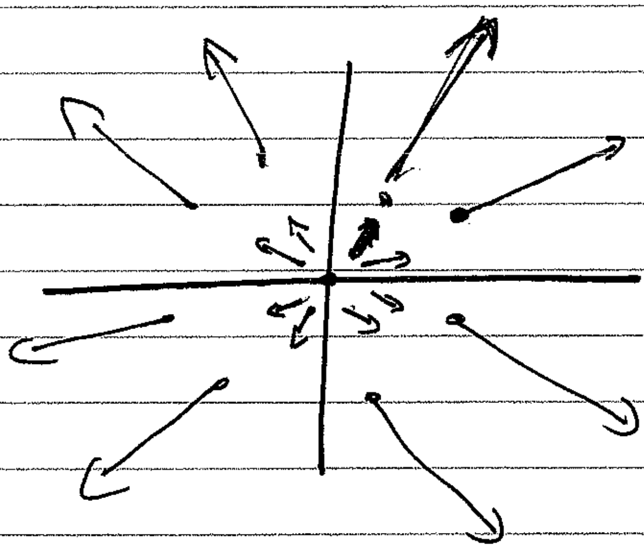
$$= \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

Ex: $F: \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x,y) = (x,y)$$

$$\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (x, y)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$



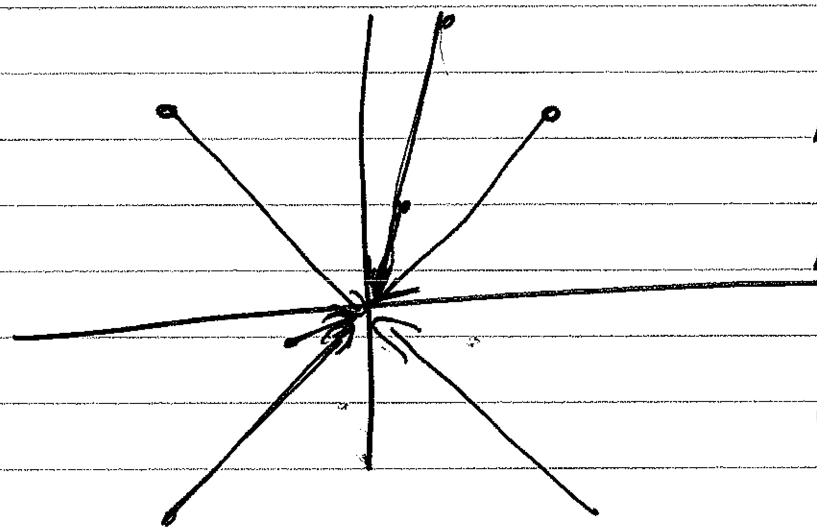
more fluid flows out ~~into~~ a point than into the point

$\nabla \cdot F > 0$

$$F(x,y) = (-x, -y)$$

$$\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (-x, -y)$$

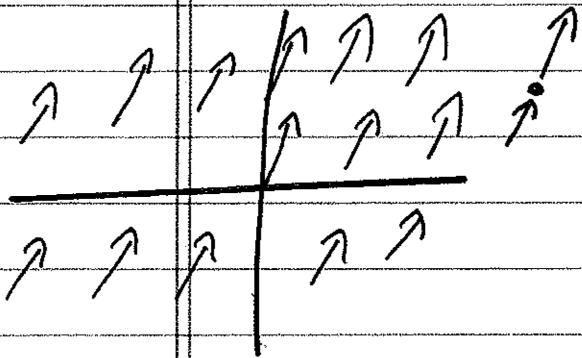
$$= \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -1 + -1 = -2$$



more fluid flowing in than flowing out

$\nabla \cdot F < 0$

$F(x,y) = (1,2)$



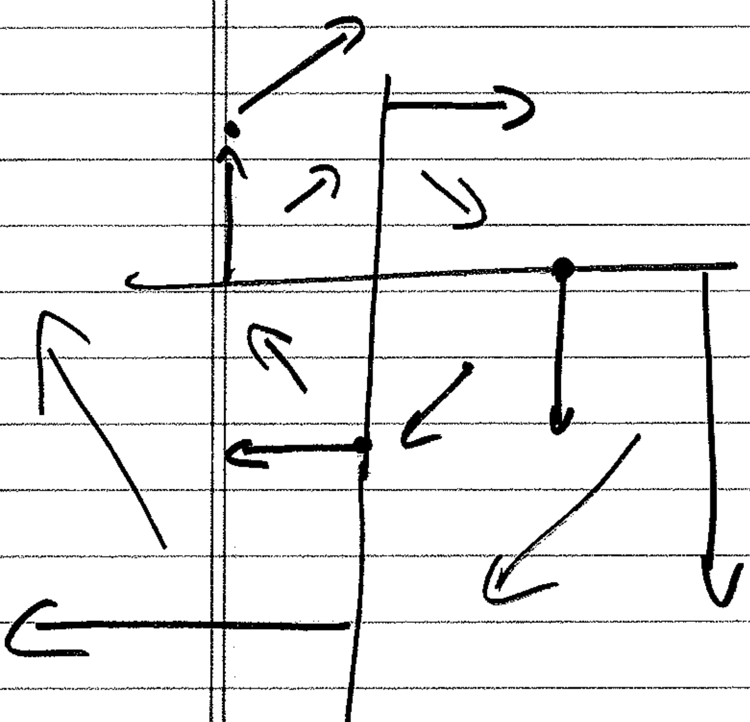
Fluid flowing in = Fluid flowing out

$\Rightarrow \nabla \cdot F = 0$

or algebraically

$$\nabla \cdot F = \frac{\partial (1)}{\partial x} + \frac{\partial (2)}{\partial y} = 0$$

$G(x,y) = (y, -x)$



Fluid flowing in = Fluid flowing out

$\Rightarrow \nabla \cdot F = 0$

OR algebraically

$$\nabla \cdot F = \frac{\partial (y)}{\partial x} + \frac{\partial (-x)}{\partial y}$$

$$= 0 + 0 = 0$$

$$F : X \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

must be 3-dim

$$\text{curl } F = \nabla \times F$$

Ex: $F(x,y,z) = (x, y, z)$

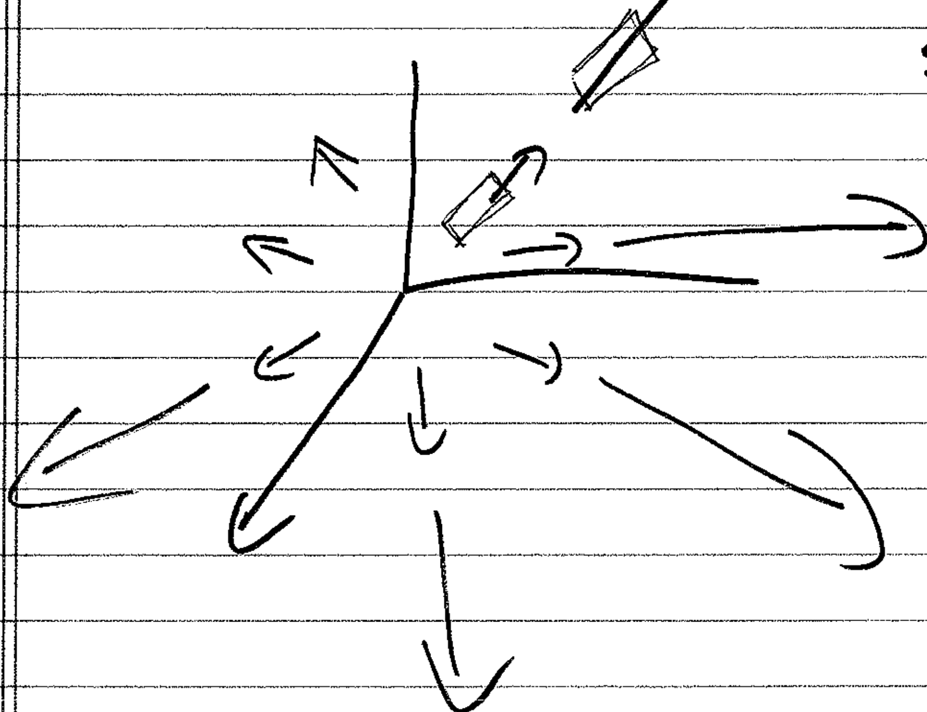
$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - j\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

not $y\frac{\partial}{\partial z}$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = (0,0,0)$$

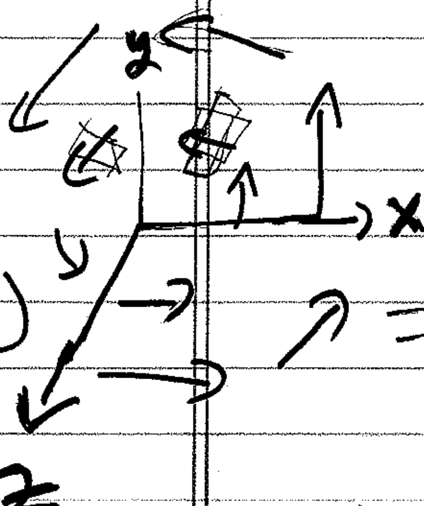
$$F(x, y, z) = (x, y, z)$$



stick thrown into water doesn't rotate

$$F(x, y, z) = (-y, x, 0)$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & +x & 0 \end{vmatrix}$$



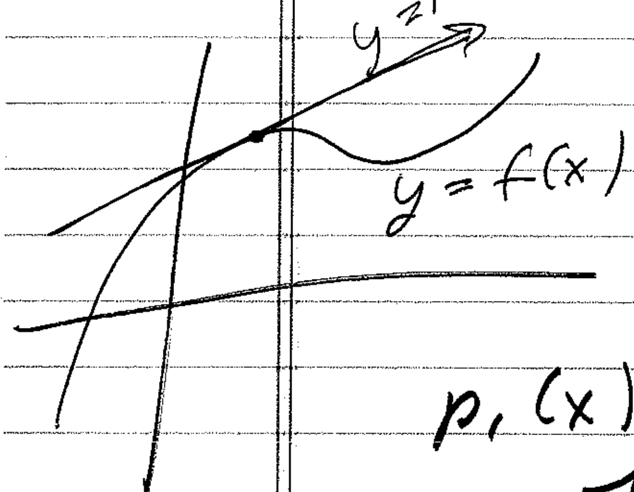
$$= \left(\frac{\partial (0)}{\partial y} - \frac{\partial (x)}{\partial z}, -(0+0), \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right)$$

$$= (0, 0, +2)$$

Ch 4:

Calc 1: $f: \mathbb{R} \to \mathbb{R}$

$$f(x) \sim \underbrace{f(a) + f'(a)(x-a)}_{\text{tangent line at } x=a} = p_1(x)$$



$p_1(x)$ is a linear approx of f near a

p_1 has the properties:

p_1 is a line

$p_1(a) = f(a)$

$p_1'(a) = f'(a)$

Can also approx f <u>near a</u>

w/ a quadratic fn $= p_2$

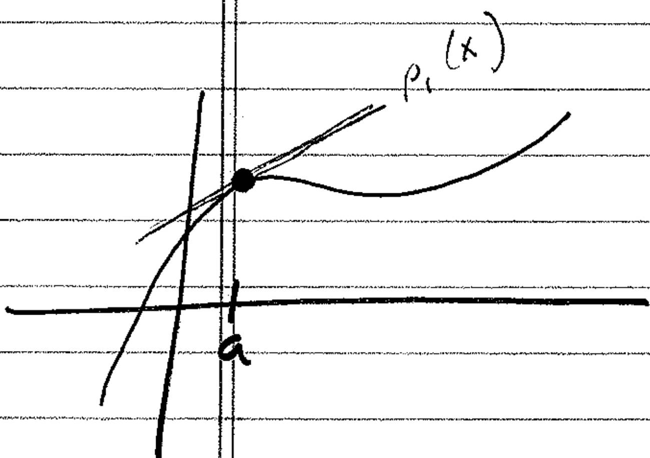
f & p_2 have same tangent at $x=a$

$$\begin{cases} p_2(a) = f(a) \\ p_2'(a) = f'(a) \end{cases}$$

line

$$\to p_1(x) = f(a) + f'(a)(x-a)$$
$$= p_2(a) + p_2'(a)(x-a)$$

Also want quadratic fn (parabala) to concave down



Want $p_2''(a) = f''(a)$

First order approx $= p_1(x) =$ tangent line

$$p_1(x) = f(a) + f'(a)(x-a)$$

(1) $p_1(a) = f(a)$ (2) $p_1'(a) = f'(a)$

2^{nd} order appox $= p_2(x) =$ polynomial of deg 2

(1) $p_2(a) = f(a)$

(2) $p_2'(a) = f'(a)$

(3) $p_2''(a) = f''(a)$

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

polynomial of deg 2

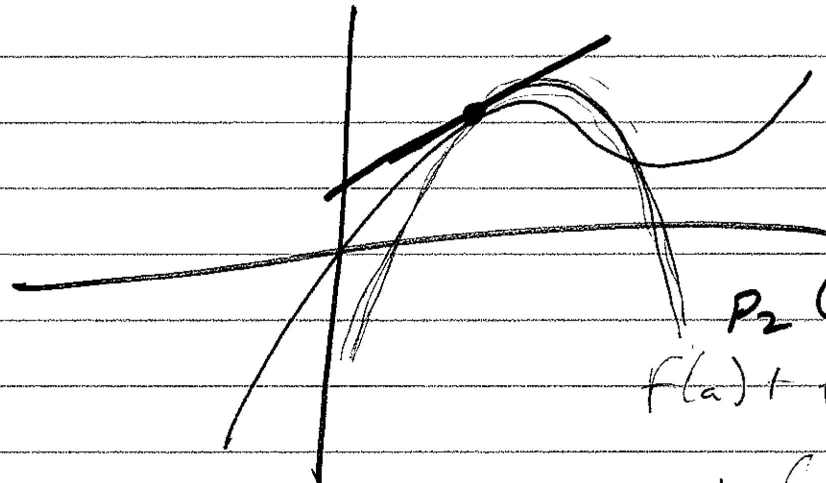
and $p_2(a) = f(a)$

$$p_2'(x) = 0 + f'(a)(1) + \cancel{2}\frac{f''(a)}{\cancel{2}}(x-a)(1)$$

$$= f'(a) + f''(a)(x-a)$$

$p_2'(a) = f'(a)$

$p_2''(x) = 0 + f''(a) = f''(a)$ so (3) holds



$P_2(x)$

$f(a) + f'(a)(x-a)$

$+ \frac{f''(a)}{2}(x-a)^2$

$P_K(x) =$ Poly of deg K approx f

$$P_K^{i}(a) = f^{i}(a)$$