2.2) $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{L}$ if

$\forall \varepsilon > 0, \ \exists \delta > 0, \ 0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow \|f(\vec{x}) - \vec{L}\| < \varepsilon$

↳ for all ↳ there exists

$$\lim_{(x,y) \to (4,5)} (2x + 3y) = 23$$

Take $\varepsilon > 0$. Let $\delta = \underline{\varepsilon / 5}$

Suppose $0 < \|(x,y) - (4,5)\| < \delta$

$\Rightarrow 0 < \sqrt{(x-4)^2 + (y-5)^2} < \delta$

$\Rightarrow \sqrt{(x-4)^2} < \delta$ and $\sqrt{(y-5)^2} < \delta$

$\Rightarrow |x-4| < \delta$ and $|y-5| < \delta$

$\Rightarrow 2|x-4| + 3|y-5| < 2\delta + 3\delta$

$\Rightarrow |2(x-4) + 3(y-5)| \le |2(x-4)| + |3(y-5)| < 5\delta$

$\Rightarrow |2(x-4) + 3(y-5)| < 5\delta = 5\left(\frac{\varepsilon}{5}\right)$

$|2(x-4) + 3(y-5)| < \varepsilon$

$|2(x-4) + 8 + 3(y-5) + 15 - 23| < \varepsilon$

Thus $|2x + 3y - 23| < \varepsilon$

$$\lim_{(x,y)\to(1,2)} \left(x^2 e^{x-2}, \ln\left(\frac{y}{2x}\right)\right)$$

since continuous, can plug in

$$= \left(1e^{-1}, \ln\left(\tfrac{2}{2(1)}\right)\right)$$

$$= \left(\tfrac{1}{e}, 0\right)$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

Need to approach $(0,0)$ all possible ways

Suppose $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0^2} = \lim_{(x,0)\to(0,0)} 1 = 1$$

Suppose $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{0^2}{0^2+y^2} = \lim 0 = 0$$

$1 \neq 0 \quad \therefore \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ DNE

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$x=0$:

$$\lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0$$

$$y=0: \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$$

How about along line $x=y$

$$x=y: \lim_{(x,x)\to(0,0)} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

$0 \neq 1/2$ so limit DNE

Note for limit $\lim_{x\to a} f(x)$ to exist, must obtain same value no matter how approach $\vec{a}$

To show $\lim_{x\to \vec{a}} f(\vec{x})$ DNE need only find 2 paths w/ different limits

To show $\lim_{x\to\vec{a}} f(\vec{x})$ exists

Much harder

but if fn continuous, then

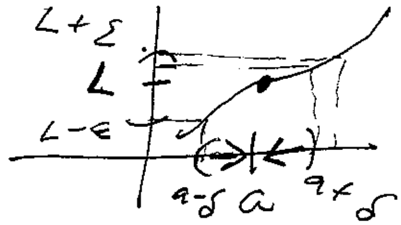
$$\lim_{x\to a} f(\vec{x}) = f(\vec{a})$$

Review calc 1

$f: \mathbb{R} \to \mathbb{R}$

open / closed

$$\lim_{x \to a} f(x) = L$$

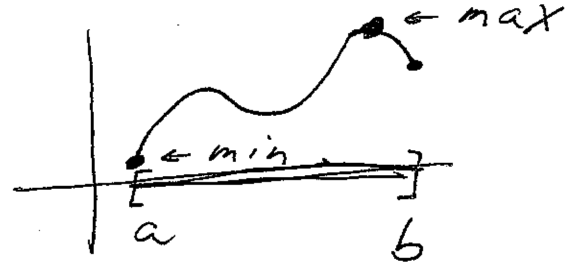


need open interval for 2-sided limit

Extreme Value Thm

$f: [a,b] \to \mathbb{R}$

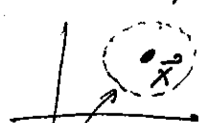
f cont on $[a,b]$



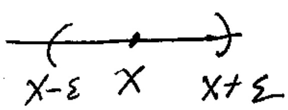
need closed interval

Need higher dimensional analogues for open/closed intervals

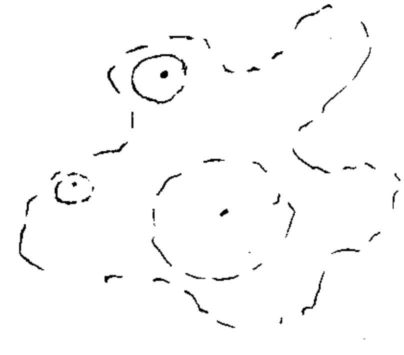
$B(\vec{x}, \varepsilon) = \{ \vec{y} \in \mathbb{R}^n \,/\, \| x - y \| < \varepsilon \}$



open ball



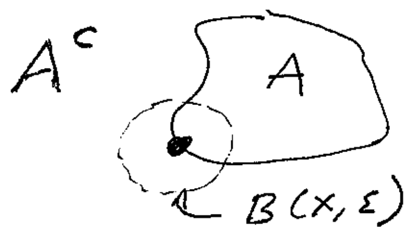
A set $U \subset \mathbb{R}^n$ is <u>open</u> if for all $x \in U$, there exists $B(x, \varepsilon) \subset U$



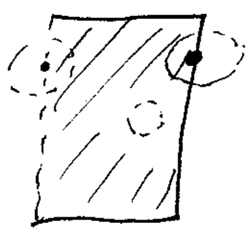
x is in the boundary of A if

$\forall \varepsilon > 0$ $\quad B(x, \varepsilon) \cap A \neq \emptyset$

$\quad B(x, \varepsilon) \cap A^c \neq \emptyset$

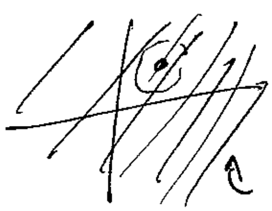


X is closed if it contains its boundary



← neither open nor closed

Unlike a door, a set can be open, closed, both open & closed (eg $\mathbb{R}^2$) or neither open nor closed



$\mathbb{R}^2$ has no boundary so $\mathbb{R}^2$ is closed

$\mathbb{R}^2$ is also open

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{L} \quad \Bigg| \quad \begin{array}{l} f: \mathbb{R}^n \to \mathbb{R}^m \\ f(\vec{x}) = (f_1(x), \dots, f_m(x)) \\ \vec{L} = (L_1, \dots, L_m) \end{array}$$

for all $\varepsilon > 0$, $\exists\, \delta > 0$ st

$$0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow \|\vec{f(\vec{x})} - \vec{L}\| < \varepsilon$$

$$\|f(\vec{x}) - \vec{L}\| = \sqrt{(f_1(\vec{x}) - L_1)^2 + \dots + (f_m(x) - L_m)^2}$$

$$> \sqrt{(f_i(\vec{x}) - L_i)^2}$$

$$= |f_i(\vec{x}) - L_i|$$

Thus

$$0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow \|f(\vec{x}) - L\| < \varepsilon$$

$$\Rightarrow |f_i(\vec{x}) - L_i| < \varepsilon$$

$$\Rightarrow \lim_{\vec{x}\to a} f_i(\vec{x}) = L_i$$