

Calculus pre-requisites you must know.

Derivative = slope of tangent line = rate.

Integral = area between curve and x-axis (where area can be negative).

The Fundamental Theorem of Calculus: Suppose  $f$  continuous on  $[a, b]$ .

1.) If  $G(x) = \int_a^x f(t)dt$ , then  $G'(x) = f(x)$ .

$$\text{I.e., } \frac{d}{dx} [\int_a^x f(t)dt] = f(x).$$

2.)  $\int_a^b f(t)dt = F(b) - F(a)$  where  $F$  is any antiderivative of  $f$ , that is  $F' = f$ .

Suppose  $f$  is cont. on  $(a, b)$  and the point  $t_0 \in (a, b)$ ,  
Solve IVP:  $\frac{dy}{dt} = f(t)$ ,  $y(t_0) = y_0$

$$dy = f(t)dt$$

$$\int dy = \int f(t)dt$$

$y = F(t) + C$  where  $F$  is any anti-derivative of  $F$ .

Initial Value Problem (IVP):  $y(t_0) = y_0$

$$y_0 = F(t_0) + C \text{ implies } C = y_0 - F(t_0)$$

Hence unique solution (if domain connected) to IVP:

$$y = F(t) + y_0 - F(t_0)$$

CH 2: Solve  $\frac{dy}{dt} = f(t, y)$

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\*\*\*1.1: Direction Fields \*\*\*\*\*

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\*\*\*\*\*Existence/Uniqueness of solution\*\*\*\*\*

Thm 2.4.2: Suppose the functions  $z = f(t, y)$  and  $z = \frac{\partial f}{\partial y}(t, y)$  are cont. on  $(a, b) \times (c, d)$  and the point  $(t_0, y_0) \in (a, b) \times (c, d)$ , then there exists an interval  $(t_0 - h, t_0 + h) \subset (a, b)$  such that there exists a unique function  $y = \phi(t)$  defined on  $(t_0 - h, t_0 + h)$  that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Thm 2.4.1: If  $p$  and  $g$  are continuous on  $(a, b)$  and the point  $t_0 \in (a, b)$ , then there exists a unique function  $y = \phi(t)$  defined on  $(a, b)$  that satisfies the following initial value problem:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0.$$

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But in general,  $y' = f(t, y)$ , solution may or may not exist and solution may or may not be unique.

Ex 1:  $y' = y' + 1$

Ex 2:  $(y')^2 = -1$

IVP ex 3:  $\frac{dy}{dx} = y(1 + \frac{1}{x}), y(0) = 1$

$\int \frac{dy}{y} = \int (1 + \frac{1}{x}) dx$  implies  $\ln|y| = x + \ln|x| + C$

$|y| = e^{x+\ln|x|+C} = e^x e^{\ln|x|} e^C = C|x|e^x = Cxe^x$

$y = \pm Cxe^x$  implies  $y = Cxe^x$

$y(0) = 1: 1 = C(0)e^0 = 0$  implies

IVP  $\frac{dy}{dx} = y(1 + \frac{1}{x}), y(0) = 1$  has no solution.

See direction field created using

[www.math.rutgers.edu/~sontag/JODE/JODEApplet.htm](http://www.math.rutgers.edu/~sontag/JODE/JODEApplet.htm)

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**Ex Non-unique:**  $y' = y^{\frac{1}{3}}$

$y = 0$  is a solution to  $y' = y^{\frac{1}{3}}$  since  $y' = 0 = 0^{\frac{1}{3}} = y^{\frac{1}{3}}$

Suppose  $y \neq 0$ . Then  $\frac{dy}{dx} = y^{\frac{1}{3}}$  implies  $y^{-\frac{1}{3}} dy = dx$

$\int y^{-\frac{1}{3}} dy = \int dx$  implies  $\frac{3}{2}y^{\frac{2}{3}} = x + C$

$y^{\frac{2}{3}} = \frac{2}{3}x + C$  implies  $y = \pm\sqrt{(\frac{2}{3}x + C)^3}$

Suppose  $y(3) = 0$ . Then  $0 = \sqrt{(2 + C)^3}$  implies  $C = -2$ . ■

Thus initial value problem,  $y' = y^{\frac{1}{3}}, y(3) = 0$ , has 3 sol'ns: ■

$y = 0, y = \sqrt{(\frac{2}{3}x - 2)^3}, y = -\sqrt{(\frac{2}{3}x - 2)^3}$

2.4 #27b. Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when  $n \neq 0, 1$  by changing it

$$y^{-n}y' + p(t)y^{1-n} = g(t)$$

when  $n \neq 0, 1$  by changing it to a linear equation by substituting  $v = y^{1-n}$

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$$\text{Solve } ty' + 2t^{-2}y = 2t^{-2}y^5$$

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Section 2.5: Solve  $\frac{dy}{dt} = f(y)$

If given either differential equation  $y' = f(y)$  OR direction field:

Find equilibrium solutions and determine if stable, unstable, semi-stable.

Understand what the above means.