

$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \leq t < 10 \\ t & t \geq 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t - 2)u_{10}(t)$

Solve $3y'' + y' + y = 2u_4(t) + (t - 2)u_{10}(t)$,
 $y(0) = 0, y'(0) = 0.$

$$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t - 2)u_{10}(t))$$

$$3[s^2\mathcal{L}(y) - sy(0) - y'(0)] + s\mathcal{L}(y) - y(0) + \mathcal{L}(y) \\ = \mathcal{L}(2u_4(t)) + \mathcal{L}((t - 10 + 8)u_{10}(t))$$

Thm: $\mathcal{L}(u_c(t)f(t - c)) = e^{-cs}\mathcal{L}(f(t)).$

If $f(t) = \underline{\hspace{2cm}}$, then $f(t - \underline{\hspace{1cm}}) = \underline{\hspace{2cm}}$

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$$3[s^2\mathcal{L}(y)] + s\mathcal{L}(y) + \mathcal{L}(y) = e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t + 8))$$

$$\mathcal{L}(y)[3s^2 + s + 1] = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$$

$$\mathcal{L}(y)[3s^2 + s + 1] = e^{-4s}\frac{2}{s} + e^{-10s}\frac{1}{s^2} + e^{-10s}\frac{8}{s}$$

$$\mathcal{L}(y) = e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}$$

$$y = 2\mathcal{L}^{-1}\left(e^{-4s}\frac{1}{s[3s^2+s+1]}\right) + \mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s^2[3s^2+s+1]}\right) \\ + 8\mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s[3s^2+s+1]}\right)$$

$$y = u_4(t)f(t - 4) + u_{10}h(t - 10) + 8u_{10}f(t - 10)$$

$$\text{where } f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right) \text{ and } h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

$$\frac{1}{s[3s^2+s+1]} = \frac{A}{s} + \frac{Bs+C}{3s^2+s+2}$$

$$1 = A(3s^2 + s + 1) + (Bs + C)s$$

$$0s^2 + 0s + 1 = (3A + B)s^2 + (A + C)s + A$$

$$0 = 3A + B, 0 = A + C, 1 = A$$

$$\text{Hence } A = 1, B = -3A = -3, C = -A = -1$$

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3\left[s^2+\frac{1}{3}s+\frac{1}{3}\right]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3\left[\left(s^2+\frac{1}{3}s+\frac{1}{3}\right)-\frac{1}{3}\right]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3\left[\left(s+\frac{1}{6}\right)^2-\frac{1}{36}+\frac{1}{3}\right]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3\left(s+\frac{1}{6}\right)}{3\left[\left(s+\frac{1}{6}\right)^2+\frac{11}{36}\right]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-\left(s+\frac{1}{6}-\frac{1}{6}+\frac{1}{3}\right)}{\left[\left(s+\frac{1}{6}\right)^2+\frac{11}{36}\right]}\right)$$

$$\begin{aligned}
&= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right) \\
&= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right) \\
&= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right) \\
&= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{\sqrt{11}}\frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right) \\
&= 1 - e^{-\frac{1}{6}t} \cos \frac{\sqrt{11}}{6}t - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t} \sin \frac{\sqrt{11}}{6}t
\end{aligned}$$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

$$\frac{1}{s^2[3s^2+s+1]} = \frac{As+D}{s^2} + \frac{Bs+C}{3s^2+s+2}$$

$$1 = (As + D)(3s^2 + s + 1) + (Bs + C)s^2$$

$$0s^3 + 0s^2 + 0s + 1 = (3A + B)s^3 + (A + 3D + C)s^2 + (A + D)s + D$$

$$0 = 3A + B, 0 = A + 3D + C, 0 = A + D, 1 = D.$$

Hence $D = 1$, $A = -D = -1$, $C = -A - 3D = 1 - 3 = -2$,
 $B = -3A = 3$.

$$\frac{1}{s^2[3s^2+s+1]} = \frac{-s+1}{s^2} + \frac{3s-2}{3s^2+s+1} = \frac{-s}{s^2} + \frac{1}{s^2} + \frac{3(s-\frac{2}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{1}{6}-\frac{2}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} = \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{5}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= -1 + t + e^{-\frac{1}{6}t} \cos \frac{\sqrt{11}}{6}t - \frac{5}{\sqrt{11}}e^{-\frac{1}{6}t} \sin \frac{\sqrt{11}}{6}t$$

Hence the final answer is

$$y = u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

$$= u_4(t)[1 - e^{-\frac{1}{6}(t-4)} \cos \frac{\sqrt{11}}{6}(t-4) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-4)} \sin \frac{\sqrt{11}}{6}(t-4)] +$$

$$u_{10}[-1 + t - 10 + e^{-\frac{1}{6}(t-10)} \cos \frac{\sqrt{11}}{6}(t-10) - \frac{5}{\sqrt{11}}e^{-\frac{1}{6}(t-10)} \sin \frac{\sqrt{11}}{6}(t-10)]$$

$$+ 8u_{10}[1 - e^{-\frac{1}{6}(t-10)} \cos \frac{\sqrt{11}}{6}(t-10) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-10)} \sin \frac{\sqrt{11}}{6}(t-10)]$$

HW due 4/8: 6.4: 2, 3, 4, 10; 6.5 #2, 5, 8, 11, 12

Exam 2: Friday April 8 over 3.9, 6.1 - 6.5